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ON THREE-DESIGNS OF SMALL ORDER

Haim HANANI*

Department of Mathematics, Technion I.I.R., Haifa, Israel

Alan HARTMAN**

IBM Scientific Research Centre, Computer Science Building, Technion City, Haifa 32000, Israel

Earl S. KRAMER***

Department of Mathematics, University of Nebraska, Lincoln, NE 68588, USA

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For positive integers $t \leq k \leq v$ and λ we define a t-design, denoted $B_t[k, \lambda; v]$, to be a pair $(X, \mathcal{B})$ where X is a set of points and $\mathcal{B}$ is a family, $(B_i : i \in I)$, of subsets of X, called blocks, which satisfy the following conditions: (i) $|X| = v$, the order of the design, (ii) $|B_i| = k$ for each $i \in I$, and (iii) every t-subset of X is contained in precisely λ blocks. The purpose of this paper is to investigate the existence of 3-designs with $3 \leq k \leq v \leq 32$ and $\lambda > 0$.

Wilson has shown that there exists a constant $N(t, k, v)$ such that designs $B_t[k, \lambda; v]$ exist provided $\lambda > N(t, k, v)$ and λ satisfies the trivial necessary conditions. We show that $N(3, k, v) = 0$ for most of the cases under consideration and we give a numerical upper bound on $N(3, k, v)$ for all $3 \leq k \leq v \leq 32$. We give explicit constructions for all the designs needed.

0. Introduction

For positive integers $t \leq k \leq v$ and λ we define a *t-design*, denoted $B_t[k, \lambda; v]$, to be a pair $(X, \mathcal{B})$ where X is a set of *points*, and $\mathcal{B}$ is a family, $(B_i : i \in I)$, of not necessarily distinct subsets of X, called *blocks*, which satisfy the following conditions

(i) $|X| = v$, the *order* of the design,
(ii) $|B_i| = k$ for each $i \in I$, and
(iii) every t-subset of X is contained in precisely λ blocks.

We denote the set of all v for which a t-design $B_t[k, \lambda; v]$ exists by $B_t(k, \lambda)$. A 2-design is often referred to as a balanced incomplete block design (BIBD) and the sets $B_2(k, \lambda)$ have been extensively investigated by Hanani [9], Wilson [16] and many others.

In this paper we investigate the existence of 3-designs with $3 \leq k \leq v \leq 32$, or in other words we attempt a description of the initial segments of the sets $B_3(k, \lambda)$.

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Let $(X, \mathscr{B})$ be a $B_t[k, \lambda; v]$ design, let I be an i-subset of X, and let b_i be the number of blocks in $\mathscr{B}$ which contain I. It is well known that

$$b_i = \lambda\binom{v-i}{t-i} \Big/ \binom{k-i}{t-i},$$

and hence the congruences

$$\lambda\binom{v-i}{t-i} \equiv 0 \quad \left(\operatorname{mod}\binom{k-i}{t-i}\right), \quad i = 0, 1, 2, \ldots, t-1 \tag{0.1}$$

are necessary for the existence of $(X, \mathscr{B})$, since each b_i must be an integer.

Wilson [17] has proved that for fixed $t \leq k \leq v$ there exists a constant $N(t, k, v)$ such that designs $B_t[k, \lambda; v]$ exist provided $\lambda > N(t, k, v)$ and λ satisfies the congruences (0.1) above. Wilson's proof gives difficult constructions of the required designs and lacks any explicit evaluation of the constant $N(t, k, v)$.

In this paper we evaluate $N(3, k, v)$ for all $v \leq 32$ and give explicit constructions of the designs needed in this evaluation.

In Section 2 we summarize some of the necessary conditions for the existence of a $B_3[k, \lambda; v]$. This enables us to give lower bounds on $N(3, k, v)$. In Section 3 we construct the designs needed to give an upper bound. Because we construct so many designs explicitly we have introduced the following notational conventions.

1. Notation

These conventions are similar to those used in Hanani [9].

Points and blocks

$I(n)$ denotes the set of non-negative integers less than n.

$Z(n)$ denotes the cyclic group of order n with $I(n)$ as the group elements and addition as the group operation.

$Z(p, x)$—only used when p is prime—denotes $Z(p)$ and indicates that x generates the multiplicative group of non-zero elements. Non-zero members of $Z(p, x)$ are denoted by their exponents, so the notation $\langle a_1, a_2, \ldots, a_n\rangle$ denotes the set of points $\{x^{a_1}, x^{a_2}, \ldots, x^{a_n}\}$. The zero of $Z(p, x)$ is written $\emptyset$.

$\mathrm{GF}(a, f(x) = 0)$—only used when q is a prime power and not prime—denotes the Galois field of order q with x as the primitive element.

As for $Z(p, x)$, the non-zero members of $\mathrm{GF}(q, f(x) = 0)$ are denoted by their exponent relative to x, and the zero is denoted $\emptyset$. Addition in $\mathrm{GF}(q, f(x) = 0)$ is denoted by the symbol $\oplus$ to distinguish it from addition of indices, which takes place in $Z(q-1)$.

When $X = Y \times Z$ sets of points are denoted $\langle(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\rangle$ where the a_i are exponents of a primitive element in Y and the b_i are exponents of a primitive element in Z.

The set $mX+c$ is defined to be $\{mx+c\colon x\in X\}$.

The brackets $\langle\ \rangle$ are used for blocks and other sets of points given by their exponents.

The words mod(q_1, q_2) after a block, B from the point set $Y\times Z$ denote the inclusion of all blocks of the form $B+(\alpha,\beta)$ (or $B\oplus(\alpha,\beta)$) with $\alpha\in Y$, $\beta\in Z$. The words mod$(q,-)$ denote the inclusion of all blocks $B+(\alpha,\langle\emptyset\rangle)$.

If X is a set of points we frequently use the shorthand $\langle(a,X)\rangle$ for the set $\{\langle a\rangle\}\times X$.

Groups

An *automorphism* of a design is a block-preserving bijection on the point set.

The group of fractional linear transformations (with non-zero determinant) $f\colon \mathrm{GF}(q)\cup\{\infty\}\to \mathrm{GF}(q)\cup\{\infty\}$,

$$f(x)=(ax+b)/(cx+d), \qquad a,b,c,d\in \mathrm{GF}(q), \quad ad-bc\neq 0$$

is denoted by PGL(q). The subgroup of PGL(q) containing all transformations with $ad-bc$ a quadratic residue in GF(q) is denoted by PSL(q). The stabilizer of ∞ in PGL(q) is the set of all transformations

$$f\colon \mathrm{GF}(q)\to \mathrm{GF}(q), \qquad f(x)=mx+c, \quad m,c\in \mathrm{GF}(q),\ m\neq 0;$$

this group is denoted AF(q). The subgroup of AF(q) containing all transformations with m a quadratic residue is denoted SAF(q).

We use the standard symbols S_n and M_n for the symmetric groups and Mathieu groups.

2. Necessary conditions

This section contains those results on non-existence of designs $B_t[k,\lambda;v]$ which we used in evaluating $N(3,k,v)$ for $v\leq 32$. We do not attempt to be encyclopaedic nor do we include the proofs of these results. A good survey of the literature in the area is given by Hedayat and Kageyama [10].

Lemma 2.1. *If* $v\in B_t(k,\lambda)$ *and* $t>i$, *then* $v-i\in B_{t-i}(k-i,\lambda)$.

Proof. If $(X,\mathcal{B})$ is a $B_t[k,\lambda;v]$ and I is an i-subset of X, then let

$$\bar{\mathcal{B}}:=\{B\backslash I \mid B\in\mathcal{B} \text{ and } I\subseteq B\}.$$

Clearly $(X\backslash I,\bar{\mathcal{B}})$ is a $B_{t-i}[k-i,\lambda;v-i]$. $\square$

Theorem 2.2 (Fisher's Inequality). *If* $v\in B_2(k,\lambda)$, *then*

$$b_0=\lambda\binom{v}{2}\Big/\binom{k}{2}\geq v. \qquad \square$$

A proof of this result may be found in Hall [7]. Using the above results we can deduce that $22 \notin B_3(7, 1)$ despite the fact that the necessary conditions (0.1) all hold.

Theorem 2.3 (Driessen [6]). *If* $\binom{n}{2}+n+1 \in B_3(n+1, 2)$, *then* $n \equiv 2$ *or* $14 \pmod{48}$. $\square$

Driessen's result is actually stronger than this, but the form above is sufficient to show that $11 \notin B_3(5, 2)$, $16 \notin B_3(6, 2)$ and $22 \notin B_3(7, 2)$.

Theorem 2.4 (Cameron [3]). *If* $v \in B_3(k, \lambda)$ *and* $\lambda\binom{v-1}{2}/\binom{k-1}{2} = v-1$, *then one of the following conditions hold*

(i) $v = 4(\lambda+1)$, $k = 2(\lambda+1)$,
(ii) $v = (\lambda+1)(\lambda^2+5\lambda+5)$, $k = (\lambda+1)(\lambda+2)$,
(iii) $v = 112$, $k = 12$, $\lambda = 1$.
(iv) $v = 496$, $k = 40$, $\lambda = 3$. $\square$

This result proves that $26 \notin B_3(10, 3)$.

3. Construction of 3-designs

We begin the construction of designs $B_3[k, \lambda; v]$ with $3 \leq k \leq v \leq 32$ by reducing the problem to the range $5 \leq k \leq \frac{1}{2}v$ and $v \leq 32$.

Lemma 3.1. *In the range* $v-t \leq k \leq v$ *we have* $v \in B_t(k, \lambda)$ *if and only if* $\lambda = c\binom{v-t}{k-t}$.

Proof. Necessity follows from (0.1), and the designs may be constructed by taking every k-subset of X precisely c times. $\square$

Wilson's paper [17] contains a proof that the designs constructed above are unique, for their parameters.

Lemma 3.2. $v \in B_t(k, \lambda)$ *if and only if* $v \in B_t(v-k, \lambda\binom{v-k}{t}/\binom{k}{t})$.

Proof. If $(X, \mathcal{B})$ is a $B_t[k, \lambda; v]$ then let $\bar{\mathcal{B}} = \{X \backslash B : B \in \mathcal{B}\}$. It is not difficult to verify that $(X, \bar{\mathcal{B}})$ is a t-design with the correct parameters. The process is reversible and this completes the proof. $\square$

The necessary conditions (0.1) on λ are equivalent to the necessary conditions on $\lambda' = \lambda\binom{v-k}{t}/\binom{k}{t}$ for block-size $k' = v-k$, so by Lemmas 3.1 and 3.2 we may restrict our study to the range $4 \leq k \leq \frac{1}{2}v$.

The case $k = 4$ has been solved by Hanani [8] where he proved the following result:

Theorem 3.3. *$v \in B_3(4, \lambda)$ if and only if the necessary conditions* (0.1) *hold.* □

This is equivalent to $N(3, 4, v) = 0$ for all $v \geq 4$.

Some of the designs constructed in [8] are defined recursively but for $v \leq 32$ an explicit description of the block sets is readily obtained.

Lemma 3.4. *If $v \in B_t(k, \lambda) \cap B_t(k, \lambda')$, then $v \in B_t(k, n\lambda + n'\lambda')$ for all non-negative integers n, n'.*

Proof. Take a disjoint union of n copies of the block set of a $B_t[k, \lambda; v]$ and n' copies of the block set of a $B_t[k, \lambda'; v]$. □

The solution to the congruence equations (0.1) for fixed v, k, t are of the form $\lambda \equiv 0 \pmod{\lambda_0}$. Lemma 3.4 gives us an explicit upper bound for $N(t, k, v)$ when we know of the existence of two designs $B_t[k, \lambda; v]$ and $B_t[k, \lambda'; v]$ with $\gcd(\lambda, \lambda') = \lambda_0$. In most cases we are able to construct a $B_t[k, \lambda_0; v]$, and thus show that $N(t, k, v) = 0$, however Lemma 3.4 is useful in the cases when $v \notin B_t(k, \lambda_0)$ or when we have not succeeded in constructing a $B_t[k, \lambda_0; v]$.

The next three results illustrate the construction of $B_3[k, \lambda; v]$ by composition methods.

Lemma 3.5. *If $v \in B_t(k, \lambda)$ and $k \in B_t(k', \lambda')$, then $v \in B_t(k', \lambda\lambda')$.*

Proof. Let $(X, \mathscr{B})$ be a $B_t[k, \lambda; v]$ and for each block $B \in \mathscr{B}$ let $(B, \mathscr{B}_B)$ be a $B_t[k', \lambda'; k]$. Taking a disjoint union of all the $\mathscr{B}_B$ gives the required design. □

Lemma 3.6. *If $v \in B_t(k, \lambda)$, then $v - 1 \in B_t(k - 1, \lambda(k - t))$.*

Proof. If $(X, \mathscr{B})$ is a $B_t[k, \lambda; v]$ and $x \in X$, then let $\mathscr{B}_x = \{B - \{x\}: B \in \mathscr{B} \text{ and } x \in B\}$. For each block $B \in \mathscr{B}$ such that $x \notin B$ let $(B, \mathscr{B}_B)$ be a $B_t[k-1, k-t; k]$ (which exists by Lemma 3.1). Taking a disjoint union of all the $\mathscr{B}_B$ and $k - t$ copies of $\mathscr{B}_x$ gives the required design. □

Lemma 3.7. *If $v - 1 \in B_t(k - 1, \lambda) \cap B_t(k, \lambda(v - k)/(k - t))$*, then

$$v \in B_t(k, \lambda(v - t)/(k - t)).$$

Proof. Let $(X, \mathscr{B})$ be a $B_t[k-1, \lambda; v-1]$, let $(X, \bar{\mathscr{B}})$ be a $B_t[k, \lambda(v-k)/(k-t); v-1]$, and let ∞ be a point not in X. Let $\mathscr{B}^* = \{B \cup \{\infty\}: B \in \mathscr{B}\} \cup \bar{\mathscr{B}}$, and it is easily verified that $(X \cup \{\infty\}, \mathscr{B}^*)$ is the required design. □

The next lemma appears to give a necessary, rather than a sufficient condition for the existence of designs. We apply it in this paper to demonstrate the existence of certain 3-designs using the existence of known designs with $t > 3$.

Lemma 3.8. *If* $v \in B_t(k, \lambda)$, *then* $v-1 \in B_{t-1}(k, \lambda(v-k)/(k-t+1))$.

Proof. Let $(X, \mathscr{B})$ be a $B_t[k, \lambda; v]$ and let ∞ be a member of X. Define $\bar{\mathscr{B}} = \{B \in \mathscr{B}: \infty \notin B\}$, then $(X \setminus \{\infty\}, \bar{\mathscr{B}})$ gives a design with the reauired parameters. □

Remark 3.9. The existence of the following t-designs has proved useful in our research

$24 \in B_5(8, 1)$,	$24 \in B_5(12, 48)$	(Witt [18]),
$24 \in B_5(9, 6)$		(Assmus and Mattson [2]),
$18 \in B_5(7, 6)$,	$22 \in B_4(8, 60)$	(Kramer [12, 13]).

We turn now to general constructions of infinite families of 3-designs.

Theorem 3.10 (Sprott [14]). *Let q be an odd prime-power, then*
(i) *if* $q \equiv 3 \pmod 4$, *then* $q+1 \in B_3(\frac{1}{2}(q+1), \frac{1}{4}(q-3))$, *or*
(ii) *if* $q \equiv 1 \pmod 4$, *then* $q+1 \in B_3(\frac{1}{2}(q+1), \frac{1}{2}(q-3))$.

Proof. Let $X = \mathrm{GF}(q) \cup \{\infty\}$, $D_0 = \langle \infty, 1, 3, 5, \ldots, q-2 \rangle$ and $D_1 = \langle \emptyset, 0, 2, 4, \ldots, q-3 \rangle$. For $q \equiv 3 \pmod 4$ take $\mathscr{B} = \{\langle D_i \rangle \bmod q, i \in I(2)\}$, then $(X, \mathscr{B})$ is the required design and it admits $\mathrm{SAF}(q)$ as automorphisms. For $q \equiv 1 \pmod 4$ let $\mathscr{B} = \{\langle i + D_j \rangle \bmod q, i \in I(2), j \in I(2)\}$, then $(X, \mathscr{B})$ is the required design and it admits $\mathrm{AF}(q)$. □

The next result is an extension theorem which assists in verifying that the configurations given above are indeed 3-designs.

Theorem 3.11 (Alltop [1]). *If* $2k-1 \in B_{2s}(k-1, \lambda)$, *then* $2k \in B_{2s+1}(k, \lambda)$.

Proof. Let $(X, \mathscr{B})$ be a $B_{2s}[k-1, \lambda; 2k-1]$ and let ∞ be a point not in X. Let $\mathscr{B}' = \{B \cup \{\infty\}: B \in \mathscr{B}\}$ and $\mathscr{B}'' = \{X - B: B \in \mathscr{B}\}$. Then $(X \cup \{\infty\}, \mathscr{B}' \cup \mathscr{B}'')$ is the required design. Verification of this is given in general by Alltop [1] and in the case $s = 1$ by Sprott [14]. □

We now give two constructions due to Hughes which include the more classical results of Witt [19] and Carmichael [4].

Theorem 3.12 (Hughes [11]). *Let q be a prime-power and let* $q \equiv 1 \pmod{k-2}$, *then*
(i) *If* $k = 4$, *then* $q+1 \in B_3(4, 3)$.

(ii) *If* $k=6$, *then* $q+1\in B_3(6,5)$.
(iii) *If* $k-1\mid q$, *then* $q+1\in B_3(k,1)$.
(iv) *Otherwise* $q+1\in B_3(k,\frac{1}{2}k(k-1))$.

Proof. Let $X=\mathrm{GF}(q)\cup\{\infty\}$, and let $\mathcal{B}$ consist of all distinct PGL(q)-images of the block

$$D_0=\left\langle\infty,\emptyset,0,\frac{q-1}{k-2},\frac{2(q-1)}{k-2},\frac{3(q-1)}{k-2},\ldots,\frac{(k-3)(q-1)}{k-2}\right\rangle.$$

This design is a 3-design since PGL(q) is 3-transitive on X. The value of λ is calculated from the size of the stabilizer of D_0 in PGL(q). □

Theorem 3.13 (Hughes [11]). *Let* q *be a prime-power and let* $q\equiv 1\pmod{k-1}$, *then*
(i) *If* $k=4$, *then* $q+1\in B_3(4,2)$.
(ii) *If* $k\mid q$, *then* $q+1\in B_3(k,k-2)$.
(iii) *Otherwise* $q+1\in B_3(k,k(k-2))$.

Proof. As above let $X=\mathrm{GF}(q)\cup\{\infty\}$ and let $\mathcal{B}$ be the set of all distinct PGL(q) images of the block

$$D_0=\left\langle\infty,0,\frac{q-1}{k-1},\frac{2(q-1)}{k-1}.\frac{3(q-1)}{k-1},\ldots,\frac{(k-2)(q-1)}{k-1}\right\rangle. \qquad \square$$

Remark 3.14. Hughes points out that the design with $q=16$ and $k=7$ in Theorem 3.12 is in fact a 4-design proving that $17\in B_4(7,6)$.

Part (iii) of Theorem 3.12 contains Witt's constructions of inversive geometries [19] and parts (i) of Theorems 3.12, 3.13 are predated by Carmichael's work on 3-designs with $k=4$ [4].

Lemma 3.15. *Let* q *be a prime-power and let* $q\equiv 1\pmod k$, *then* $q+1\in B_3(k,\frac{1}{2}(k-1)(k-2))$.

Proof. Let $X=\mathrm{GF}(q)\cup\{\infty\}$. and let $\mathcal{B}$ be the set of all distinct PGL(q)-images of the block

$$D_0=\left\langle 0,\frac{q-1}{k},\frac{2(q-1)}{k},\ldots,\frac{(k-1)(q-1)}{k}\right\rangle.$$

The stabilizer of D_0 contains a group of order $2k$ generated by the transformations $f(x)=\langle(q-1)/k\rangle x$ and $g(x)=x^{-1}$. Hence the number of blocks in $\mathcal{B}$ is a divisor of $|\mathrm{PGL}(q)|/2k=q(q^2-1)/2k$. The result then follows from the 3-transitivity of PGL(q) and Lemma 3.4. □

We now tabulate, for all $v \leq 32$, the values $5 \leq k \leq \frac{1}{2}v$ and the smallest value λ_0 for which the conditions (0.1) are satisfied. If $v \notin B_3(k, \lambda_0)$ or if we are unable to construct a $B_3[k, \lambda_0; v]$, then we also list the multiples of λ_0 required to give the bounds on $N(3, k, v)$. The groups G given in Table 3.18 are contained in the automorphism group of the designs, but are not necessarily the full automorphism group.

We illustrate the use of the table for constructing designs by the following examples

Example 3.16. $v = 14$, $k = 5$, $\lambda_0 = 5$.

$$X = Z(2, 1) \times Z(7, 3), \qquad G = \mathrm{AF}(2) \times \mathrm{AF}(7).$$

$$\begin{aligned}\mathcal{B} = {} & \{\langle(\emptyset, i)(\emptyset, i+3)(0, j)(0, j+2)(0, j+4)\rangle \bmod (2, 7), i \in I(3), j \in I(2)\} \\ & \cup \{\langle(\emptyset, i+1)(\emptyset, i+4)(0, \emptyset)(0, i)(0, i+3)\rangle \bmod (2, 7), i \in I(3)\} \\ & \cup \{\langle(\emptyset, \emptyset)(0, \emptyset)(0, j)(0, j+2)(0, j+4)\rangle \bmod (2, 7), j \in I(2)\} \text{ (twice).}\end{aligned}$$

The first base block has a full orbit of size 2.7.6 under the action of AF(2)× AF(7). The second base block is stabilized by the permutation $\langle(i, j)\rangle \to \langle(i, 3+j)\rangle$ and so has an orbit of size 2.7.3. The third base block has an orbit of length 2.7.2 since $\langle(i, j)\rangle \to \langle(i, 2+j)\rangle$ is in its stabilizer. □

Example 3.17. $v = 32$, $k = 6$, $\lambda = 6 = 3\lambda_0$.

$$X = \mathrm{GF}(32, x^5 = x^3 + 1), \qquad G = \mathrm{AF}(32).$$

$$\langle\emptyset, D_0\rangle = \langle\emptyset, 0, 2, 4, 25, 28\rangle = \{0, 1, x^2, x^4, x^4+1, x^2+1\},$$

$$D_1 = \langle 0, 1, 2, 3, 4, 14\rangle.$$

$$\begin{aligned}\mathcal{B} = {} & \{\langle\langle a\langle 4\rangle \oplus b\langle 3\rangle \oplus c\langle 2\rangle \oplus d\langle 1\rangle \oplus \langle\emptyset, D_0\rangle\rangle + i\rangle, a, b, c, d \in I(2), i \in I(31)\} \\ & \cup \{\langle i + D_1\rangle \bmod 32, i \in I(31)\}.\end{aligned}$$

The first base block $\langle\emptyset, D_0\rangle$ is stabilized by the permutation $\langle j\rangle \to \langle j\rangle \oplus \langle 0\rangle$, and thus has an orbit of length 16.31. The second base block has full orbit of size 32.31.

The existence of a stabilizer in the additive subgroup of AF(p^n) is not clear from the notation. Rewriting the blocks in standard notation clarifies this situation. □

The authors would gratefully acknowledge any additions or corrections to Table 3.18. The methods used for construction and verification of the designs listed is described in [12], although the techniques are quite standard.

Table 3.18. Construction of $B_3[k, \lambda; v]$ for $v \geq 32$, $5 \leq k \leq v/2$.
The block fragment D_i always refers to the most recently defined D_i.

v	k	λ	Construction
10	5	3	Theorem 3.10
11	5	2	Does not exist by Theorem 2.3.
11	5	4	$X = Z(11, 2)$, $G = \mathrm{SAF}(11)$, $\beta = \{\langle 2i + I(5)\rangle \bmod 11, i \in I(5)\} \cup \{\langle 0, 2, 4, 6, 8\rangle \bmod 11\}$.
11	5	6	Lemma 3.6, $12 \in B_3(6, 2)$—below.
12	5	6	Lemma 3.5, $12 \in B_3(6, 2)$—below, $6 \in B_3(5, 3)$—Lemma 3.1.
12	6	2	Theorem 3.10.
13	5	15	Lemma 3.6, $14 \in B_3(6, 5)$—below.
13	6	20	Lemma 3.6, $14 \in B_3(7, 5)$—below.
14	5	5	$X = Z(2, 1) \times Z(7, 3)$, $G = \mathrm{AF}(2) \times \mathrm{AF}(7)$, $D_0 = \langle 0, 3\rangle$, $D_1 = \langle 0, 2, 4\rangle$, $\mathcal{B} = \{\langle(\emptyset, i + D_0), (0, i + D_1)\rangle \bmod(2, 7), i \in I(6)\}$ $\cup \{\langle(\emptyset, i + 1 + D_0), (0, \emptyset), (0, i + D_0)\rangle \bmod(2, 7), i \in I(3)\}$ $\cup \{\langle(\emptyset, \emptyset), (0, \emptyset), (0, j + D_1)\rangle \bmod(2, 7), j \in I(2)\}$ (*twice*).
14	6	5	Theorem 3.12.
14	7	5	Theorem 3.10.
15	5	6	Lemma 3.7, $14 \in B_3(4, 1)$—Theorem 3.3, $14 \in B_3(5, 5)$—above.
15	6	20	Lemma 3.7, $14 \in B_3(5, 5) \cap B_3(6, 5)$—above.
15	7	15	Lemma 3.6, $16 \in B_3(8, 3)$—below.
16	5	6	From Kramer [12] we let $X = Z(17, 3) \cup \{\infty\}$ and $G = \mathrm{PGL}(17)$. If $B = \langle\infty, 0, 2, 4, 8, 10, 12\rangle$, then B is stabilized by $\alpha \to \langle 8\rangle\alpha$ and B lies in an orbit $\mathcal{B}$ of length 2448. $\mathcal{B}$ is a $B_5[7, 6; 18]$ and Lemma 2.1 yields our $B_3[5, 6; 16]$.
16	6	2	Does not exist by Theorem 2.3.
16	6	4	$X = \mathrm{GF}(16, x^4 = x + 1)$, $G = \mathrm{AF}(16)/\langle\alpha \to \langle 3\rangle\alpha\rangle$, $D_0 = \langle 0, 1, 2, 6, 7\rangle$, $D_1 = \langle 3I(5)\rangle$, $\mathcal{B} = \{\langle\emptyset, 3i + D_0\rangle \bmod 16, i \in I(5)\} \cup \{\langle\emptyset, j + D_1\rangle \bmod 16, j \in I(2)\}$.
16	6	6	Lemma 2.1, $17 \in B_4(7, 6)$—Remark 3.14.
16	7	5	Unknown.
16	7	10	$X = Z(2, 1) \times Z(7, 3) \cup \{\infty_0, \infty_1\}$, $G = \mathrm{AF}(2) \times \mathrm{SAF}(7)$, $D_0 = \langle 0, 2, 4\rangle$, $D_1 = \langle 0, 3\rangle$, $D_2 = \langle 0, 2, 3, 5\rangle$, $\mathcal{B} = \{\langle\infty_0, \infty_1, (\emptyset, \emptyset), (0, \emptyset), (0, j + D_0)\rangle \bmod(2, 7), j \in I(2)\}$ $\cup \{\langle\infty_0, (\emptyset, D_0), (0, 2i + 3 + I(3))\rangle \bmod(2, 7), i \in I(3)\}$ $\cup \{\langle\infty_1, (\emptyset, 1 + D_0), (0, 2i + I(3))\rangle \bmod(2, 7), i \in I(3)\}$ $\cup \{\langle(\emptyset, \emptyset), (\emptyset, D_1), (0, i + D_2)\rangle \bmod(2, 7), i \in I(3)\}$ $\cup \{\langle(\emptyset, \emptyset), (\emptyset, I(6))\rangle \bmod(2, -)\}$ (three times).
16	7	15	Lemma 3.5, $16 \in B_3(8, 3)$—below, $8 \in B_3(7, 5)$—Lemma 3.1,
16	8	3	Theorem 3.11, $15 \in B_2(7, 3)$—Hanani [9].
17	5	1	Theorem 3.12.
17	6	4	$X = Z(17, 3)$, $G = \mathrm{AF}(17)$, $D_0 = \langle 0, 1, 3\rangle$, $\mathcal{B} = \{\langle i + D_0, i + 8 + D_0\rangle \bmod 17, i \in I(8)\}$.
17	7	7	Unknown.
17	7	14	Lemma 3.7, $16 \in B_3(6, 4) \cap B_3(7, 10)$—above.
17	7	21	Theorem 3.12.
17	8	14	$X = Z(17, 3)$, $G = \mathrm{AF}(17)$, $D_0 = \langle 0, 1, 3, 5\rangle$, $D_1 = \langle 2I(8)\rangle$, $\mathcal{B} = \{\langle i + D_0, i + 8 + D_0\rangle \bmod 17, i \in I(8)\} \cup \{\langle j + D_1\rangle \bmod 17, j \in I(2)\}$.
18	5	15	Theorem 3.13.
18	6	5	Theorem 3.12.
18	7	105	Lemma 3.5, $18 \in B_3(8, 21)$—below, $8 \in B_3(7, 5)$—Lemma 3.1.
18	8	21	Lemma 3.15.
18	9	7	Theorem 3.10.
19	5	30	Lemma 3.6, $20 \in B_3(6, 10)$—below.

Table 3.18. (Contd.)

v	k	λ	Construction
19	6	20	$X=Z(19,2)$, $G=AF(19)$, $D_0=\langle 0,1,3\rangle$ $D_1=\langle j+6I(3); j=0,1\rangle$, $D_2=\langle j+6I(3); j=0,2\rangle$. $\mathcal{B}=\{\langle i+D_0, i+9+D_0\rangle \bmod 19, i\in I(9)\}$ (three times) $\cup\{\langle j+D_1\rangle \bmod 19, j\in I(6)\}$ (three times) $\cup\{\langle j+D_2\rangle \bmod 19, j\in I(6)\}$.
19	7	35	$X=Z(19,2)$, $G=AF(19)$, $D_0=\langle j+9I(2): j=0,1,3\rangle$, $D_1=\langle j+9I(2): j=0,2,4\rangle$, $D_2=\langle j+6I(3): j=0,2\rangle$, $\mathcal{B}=\{\langle\emptyset, i+D_0\rangle \bmod 19, i\in I(9)\}$ (four times) $\cup\{\langle\emptyset, i+D_1\rangle \bmod 19, i\in I(9)\}\cup\{\langle\emptyset, i+D_2\rangle \bmod 19, i\in I(6)\}$.
19	8	168	Lemma 3.6. $20\in B_3(9,28)$—below.
19	9	28	Lemma 3.6, $20\in B_3(10,4)$—below.
20	5	2	Unknown.
20	5	4	$X=(Z(3,2)\cup\{\infty\})\times Z(5,2)$, $G=AF(3)\times AF(5)$, $D_0=\langle 0,2\rangle$, $\mathcal{B}=\{\langle(\infty,\emptyset),(\infty,I(4))\rangle\}$ $\cup\{\langle(\infty,\emptyset),(\infty,j+D_0),(\emptyset,j+1+D_0)\rangle \bmod(3,5), j\in I(2)\}$ $\cup\{\langle(\infty,i+D_0),(j,i+D_0),(j+1,i+1)\rangle \bmod(3,5), i\in I(4), j\in I(2)\}$ $\cup\{\langle(\infty,i+D_0),(\emptyset,\emptyset),(I(2),\emptyset)\rangle \bmod(-,5), i\in I(2)\}$ (twice) $\cup\{\langle(\infty,\emptyset),(\emptyset,i+I(2)),(j,i),(j+1,i+3)\rangle \bmod(3,5), i\in I(4), j\in I(2)\}$ $\cup\{\langle(\infty,\emptyset),(\emptyset,i+D_0),(I(2),\emptyset)\rangle \bmod(3,5), i\in I(2)\}$ (twice) $\cup\{\langle(\infty,\emptyset),(0,I(4))\rangle \bmod(3,5)\}$ $\cup\{\langle(i,\emptyset),(i,j+D_0),(i+1,j+D_0)\rangle \bmod(3,5), i\in I(2), j\in I(2)\}$ $\cup\{\langle(\emptyset,\emptyset),(i,D_0),(i+1,1+D_0)\rangle \bmod(3,5), i\in I(2)\}$.
20	5	6	$X=Z(19,2)\cup\{\infty\}$, $G=AF(19)$ $D_0=\langle 0,3,9,12\rangle$, $D_1=\langle 0,1,3,9\rangle$. $\mathcal{B}=\{\langle\infty, i+D_0\rangle \bmod 19, i\in I(9)\}\cup\{\langle\emptyset, j+D_1\rangle \bmod 19,$ $j\in I(18)\}\cup\{\langle\emptyset, i+D_0\rangle \bmod 19, i\in I(9)\}$.
20	6	10	Lemma 3.15.
20	7	35	Theorem 3.13.
20	8	14	$X=Z(19,2)\cup\{\infty\}$, $G=AF(19)$, $D_0=\langle j+6I(3): j=0,2\rangle$, $D_1=\langle 0,1,4,6\rangle$. $\mathcal{B}=\{\langle\infty,\emptyset, i+D_0\rangle \bmod 19, i\in I(6)\}\cup\{\langle i+D_1, i+9+D_1\rangle \bmod 19, i\in I(9)\}$,
20	9	28	Lemma 3.15.
20	10	4	Theorem 3.10.
21	5	3	Lemma 3.6, $22\in B_3(6,1)$—below.
21	6	4	Unknown.
21	6	8	$X=\mathcal{P}_2(I(7))$, $G=S_7$. X is viewed as the 21 edges of K_7, the complete graph on 7 vertices. $\mathcal{B}_0=\{\mathcal{P}_2(S): \|S\|=4, S\subseteq I(7)\}$ is the set of all 35 complete 4-vertex (with 6 edges) subgraphs of K_7. $\mathcal{B}_1=\{\{a,x_0\},\ldots,\{a,x_5\}: S=\{a,x_0,\ldots,x_5\}\subseteq I(7), \|S\|=7\}$ is the set of 7 distinct 6-edge star-shaped subgraphs of K_7. $\mathcal{B}_2=\{\{x_0,x_1\},\{x_1,x_2\},\ldots,\{x_4,x_5\},\{x_5,x_0\}: S=\{x_0,\ldots,x_5\}\subseteq I(7), \|S\|=6\}$ is the set of 420 distinct 6-cycle subgraphs of K_7. $\mathcal{B}=\mathcal{B}_0$ (twice) $\cup\mathcal{B}_1$ (six times) $\cup\mathcal{B}_2$.
21	6	12	$X=Z(19,2)\cup\{\infty_0,\infty_1\}$, $G=SAF(19)$, $D_i=\langle 0,1,5,11+2i\rangle, i\in I(2)$, $D_2=\langle 6I(3)\rangle$, $\mathcal{B}=\{\langle\infty_0,\infty_1,\emptyset, 2i+D_2\rangle \bmod 19, i\in I(3)\}$ $\cup\{\langle 2i+D_2, 2i+2+D_2\rangle \bmod 19, i\in I(3)\}$ $\cup\{\langle\infty_i,\emptyset, 2j+i+D_i\rangle \bmod 19, i\in I(2), j\in I(9)\}$ $\cup\{\langle\emptyset, j+I(5)\rangle \bmod 19, j\in I(18)\}$.
21	7	15	Unknown.
21	[illegible]	30	Unknown.

Table 3.18. (Contd.)

v	k	λ	Construction
21	7	45	$X = Z(19,2)\cup\{\infty_0,\infty_1\}$ $G = \mathrm{SAF}(19)$, $D_0 = \langle j+6I(3): j=0,2\rangle$, $D_1 = \langle 1+D_0\rangle$, $D_2 = \langle 3I(6)\rangle$. $D_3 = \langle 0,1,3,5\rangle$, $D_{4+i} = \langle 0,1,2,3,4+4i\rangle$, $i\in I(2)$, $D_{6+i} = \langle I(5), 7+6i\rangle$, $i \in I(2)$. $\mathscr{B} = \{\langle\infty_i, 2j+D_i\rangle \bmod 19, j\in I(3), i\in I(2)\}$ $\cup\{\langle\emptyset, i+D_2\rangle \bmod 19, i\in I(3)\}\cup\{\langle\infty_0,\infty_1,\emptyset, 2i+D_3\rangle \bmod 19, i\in I(9)\}$ $\cup\{\langle\infty_i,\emptyset, j+D_{4+i}\rangle \bmod 19, j\in I(18), i\in I(2)\}$ $\cup\{\langle\emptyset, j+D_{6+i}\rangle \bmod 19, j\in I(18), i\in I(2)\}$.
21	7	60	$X=\{2,3,\ldots,22\}$, $G = \mathrm{PSL}(3,4)$. Let M_{24} be represented as in Todd [15]. Then our G is the subgroup of M_{24} which fixes each of the three points ∞, 0, 1. Our G (of order 20, 160) can be generated by the permutations $\alpha = (2)(3,14)(4,6,22,7)(5,17)(8,9,18,10)(11,15,12,20)(13,19,16,21)$ and $\beta = (2,16,9,6,8)(3,12,13,18,4)(5)(7,17,10,11,22)(14,19,21,20,15)$. Under G the block $D_0 = \{2,3,4,5,8,11,13\}$ is stabilized by a subgroup of order 168 and so lies in an orbit $\mathscr{B}_0$ of 120 blocks. The block $D_1 = \{2,3,4,5,6,16,21\}$ lies in an orbit $\mathscr{B}_1$ of 840 blocks. Our required design has blocks $\mathscr{B} = \mathscr{B}_0$ (twelve times) $\cup\mathscr{B}_1$.
21	7	75	Unknown.
21	8	84	Unknown.
21	8	168	$X=\{2,3,\ldots,22\}$, $G = \mathrm{PSL}(3,4)$. Our G is exactly the same as the G we used in showing $21\in B_3(7,60)$. Under G the block $D_1 = \{2,3,4,5,6,9,12,14\}$ lies in an orbit $\mathscr{B}_1$ of 210 blocks and $D_2 = \{2,3,4,5,6,7,17,22\}$ lies in an orbit $\mathscr{B}_2$ of 1680 blocks. Our required design has blocks $\mathscr{B} = \mathscr{B}_1$ (eleven times) $\cup\mathscr{B}_2$.
21	8	252	Lemma 3.6, $22\in B_3(9,42)$—below.
21	9	42	Lemma 3.7, $20\in B_3(8,14)\cap B_3(9,28)$—above.
21	10	72	Lemma 3.6, $22\in B_3(11,9)$—below.
22	5	3	Lemma 3.5, $22\in B_3(6,1)$—below, $6\in B_3(5,3)$—Lemma 3.1.
22	6	1	Lemma 2.1, $24\in B_5(8,1)$—Remark 3.9, see also $21\in B_3(7,60)$—above.
22	7	1	Does not exist by Lemma 2.1 and Theorem 2.2.
22	7	2	Does not exist by Theorem 2.3.
22	7	3	Unknown.
22	7	4	Apply Lemma 2.1 and Lemma 3.8 to the $B_5[8,1,24]$ which exists by Remark 3.9, see also $21\in B_3(7,60)$—above.
22	7	5	$X = Z(2,1)\times Z(11,2)$, $G = \mathrm{AF}(2)\times\mathrm{AF}(11)$, $D_0 = \langle 0,2,5,7\rangle$, $D_1 = \langle 4,9\rangle$, $D_2 = \langle 0,5\rangle$, $D_3 = \langle 0,1,5,6\rangle$, $\mathscr{B} = \{\langle(\emptyset,\emptyset),(\emptyset, i+D_0),(0,i+D_1)\rangle \bmod(2,11), i\in I(5)\}$ $\cup\{\langle(\emptyset,\emptyset),(\emptyset,i+D_2),(0,i+D_3)\rangle \bmod(2,11), i\in I(5)\}$.
22	7	6	$X = Z(2,1)\times Z(11,2)$, $G = \mathrm{AF}(2)\times\mathrm{AF}(11)$, $D_0 = \langle 0,2,4,6,8\rangle$, $D_1 = \langle 3,9\rangle$, $D_2 = \langle 0,1,2\rangle$, $\mathscr{B} = \{\langle(\emptyset,\emptyset),(0,\emptyset),(0,i+D_0)\rangle \bmod(2,11), i\in I(2)\}$ $\cup\{\langle(\emptyset,\emptyset)(\emptyset,i+D_1),(0,\emptyset),(0,i+D_2)\rangle \bmod(2,11), i\in I(10)\}$.
22	7	7	Unknown.
22	8	6	Unknown.
22	8	12	Apply Lemma 3.8 twice to the $B_5[8,1;24]$ which exists by Remark 3.9 see also $21\in B_3(7,60)$ above.
22	8	18	$X = Z(2,1)\times Z(11,2)$, $G = \mathrm{AF}(2)\times\mathrm{AF}(11)$, $D_0 = \langle 0,2,5,7\rangle$, $D_1 = \langle 0,1,5,6\rangle$, $D_2 = \langle 4,9\rangle$, $D_3 = \langle 0,1,3,5,6,8\rangle$, $D_4 = \langle 6,8\rangle$, $D_5 = \langle 0,1,2,7\rangle$, $\mathscr{B} = \{\langle(\emptyset, i+D_0),(0,i+D_0)\rangle \bmod(-,11), i\in I(5)\}$ $\cup\{\langle(\emptyset,i+D_1),(0,i+1+D_1)\rangle \bmod(2,11), i\in I(5)\}$ $\cup\{\langle(\emptyset,i+D_2),(0,i+D_3)\rangle \bmod(2,11), i\in I(5)\}$

Table 3.18. (Contd.)

v	k	λ	Construction
			$\cup\{\langle(\emptyset,\emptyset),(\emptyset,i+D_4),(0,\emptyset),(0,i+D_5)\rangle \bmod(2,11),\ i\in I(10)\}$.
22	9	42	$X=Z(2,1)\times Z(11,2)$, $G=\mathrm{AF}(2)\times\mathrm{AF}(11)$, $D_0=\langle 0,1,2,5,6,7\rangle$, $D_1=\langle 0,5\rangle$, $D_2=\langle 0,1,5,6\rangle$, $D_3=\langle 1,4,6,9\rangle$, $D_4=\langle 0,1,3,5,6,8\rangle$, $\mathscr{B}=\{\langle(\emptyset,i+D_0),(0,\emptyset),(0,i+j+3+D_1)\rangle \bmod(2,11),\ i\in I(5),\ j\in I(2)\}$ $\cup\{\langle(\emptyset,\emptyset),(\emptyset,i+D_2),(0,i+D_2)\rangle \bmod(2,11),\ i\in I(5)\}$ (twice) $\cup\{\langle(\emptyset,\emptyset),(\emptyset,i+D_2),(0,i+D_3)\rangle \bmod(2,11),\ i\in I(5)\}$ $\cup\{\langle(\emptyset,i+D_0),(0,\emptyset),(0,i+D_1)\rangle \bmod(2,11),\ i\in I(5)\}$ $\cup\{\langle(\emptyset,i+D_4),(0,\emptyset),(0,i+2+D_1)\rangle \bmod(2,11),\ i\in I(5)\}$. (also Lemma 3.8 applied twice to $B_5[9,6;24]$—Remark 3.9).
22	10	$6i$	$i=1,2,3,4,5,6$ unknown.
22	10	42	$X=Z(2,1)\times Z(11,2)$, $G=\mathrm{AF}(2)\times\mathrm{AF}(11)$ $D_0=\langle 0,2,4,6,8\rangle$, $D_1=\langle 0,1,5,6\rangle$, $D_2=\langle 2,4,6,9\rangle$, $D_3=\langle 1,4,7,9\rangle$, $D_4=\langle 0,1,4,7,8\rangle$, $D_5=\langle 2,3,5,6,9\rangle$, $D_6=\langle 0,3,5,8\rangle$, $D_7=\langle 0,1,2,5,6,7\rangle$, $D_8=\langle 0,2,5,7\rangle$, $D_9=\langle 0,1,4,5,6,9\rangle$, $\mathscr{B}=\{\langle(\emptyset,I(10))\rangle \bmod(2,11)\}\cup\{\langle(\emptyset,i+D_0),(0,i+D_0)\rangle \bmod(-,11),\ i\in I(2)\}$ $\cup\{\langle(\emptyset,\emptyset),(0,\emptyset),(\emptyset,i+D_1),(0,i+D_1)\rangle \bmod(-,11),\ i\in I(5)\}$ $\cup\{\langle(\emptyset,\emptyset),(0,\emptyset),(\emptyset,i+D_2),(0,i+D_3)\rangle \bmod(-,11),\ i\in I(10)\}$ $\cup\{\langle(\emptyset,i+D_4),(0,i+D_5)\rangle \bmod(-,11),\ i\in I(10)\}$ $\cup\{\langle(\emptyset,i+D_j),(0,i+D_{j+1})\rangle \bmod(2,11),\ i\in(5),\ j=6,8\}$.
22	10	48	$X=Z(2,1)\times Z(11,2)$, $G=\mathrm{AF}(2)\times\mathrm{AF}(11)$, $D_{10}=\langle 0,1,5,6\rangle$, $D_{11}=\langle 3,8\rangle$, $D_{12}=\langle 0,2,3,5,7,8\rangle$, $D_{13}=\langle 0,4,5,9\rangle$, $\mathscr{B}=\{\langle(\emptyset,I(10))\rangle \bmod(2,11)\}$ $\cup\{\langle(\emptyset,i+D_0),(0,i+D_0)\rangle \bmod(-,11),\ i\in I(2)\}$ (four times) $\cup\{\langle(\emptyset,i+D_0),(0,i+1+D_0)\rangle \bmod(-,11),\ i\in I(2)\}$ (three times) $\cup\{\langle(\emptyset,i+D_4),(0,i+D_5)\rangle \bmod(-,11),\ i\in I(10)\}$ $\cup\{\langle(\emptyset,i+D_j),(0,i+D_{10}),(0,i+D_{11})\rangle \bmod(2,11),\ i\in I(5),\ j=10,13\}$ $\cup\{\langle(\emptyset,i+D_{10}),(0,i+D_{12})\rangle \bmod(2,11),\ i\in I(5)\}$. (see also Lemma 2.1 applied to $B_5[12,48;24]$—Remark 3.9.
22	10	54	$X=Z(2,1)\times Z(11,2)$, $G=\mathrm{AF}(2)\times\mathrm{AF}(11)$, $D_{14}=\langle 2,3,4,7,8,9\rangle$, $\mathscr{B}=\{\langle(\emptyset,I(10))\rangle \bmod(2,11)\}$ $\cup\{\langle(\emptyset,i+D_0),(0,i+D_0)\rangle \bmod(-,11),\ i\in I(2)\}$ (three times) $\cup\{\langle(\emptyset,\emptyset),(0,\emptyset),(\emptyset,i+D_1),(0,i+D_1)\rangle \bmod(-,11),\ i\in I(5)\}$ $\cup\{\langle(\emptyset,i+D_4),(0,i+D_5)\rangle \bmod(-,11),\ i\in I(10)\}$ $\cup\{\langle(\emptyset,i+D_6),(0,i+D_7)\rangle \bmod(2,11),\ i\in I(5)\}$ $\cup\{\langle(\emptyset,i+D_{10}),(0,i+D_{14})\rangle \bmod(2,11),\ i\in I(5)\}$ $\cup\{\langle(\emptyset,i+D_8),(0,i+D_{11}),(0,i+D_{13})\rangle \bmod(2,11),\ i\in I(5)\}$ $\cup\{\langle(\emptyset,i+D_{10}),(0,i+D_{10}),(0,i+D_{11})\rangle \bmod(2,11),\ i\in I(5)\}$.
22	10	60	Unknown.
22	10	66	Unknown.
22	10	72	Lemma 3.5, $22\in B_3(11,9)$—below, $11\in B_3(10,8)$—Lemma 3.1.
22	10	78	Unknown.
22	11	9	Theorem 3.11, $22\in B_2(10,9)$—see Hanani [9].
23	5	10	Lemma 3.7, $22\in B_3(4,1)$—Theorem 3.3, $22\in B_3(5,3)$—above.
23	6	20	Lemma 3.7, $22\in B_3(5,3)\cap B_3(6,1)$—above.
23	7	5	Apply Lemma 2.1 to $B_5[8,1;24]$ (see $21\in B_3(7,60)$—above) to get $B_4[7,1;23]=B_3[7,5;23]$.
23	8	8	Unknown.
23	8	16	Lemma 3.7, $22\in B_3(7,4)\cap B_3(8,12)$—above.

Table 3.18. (Contd.)

v	k	λ	Construction
23	8	24	Lemma 3.7, $22 \in B_3(7,6) \cap B_3(8,18)$—above.
23	9	12	Unknown.
23	9	24	$X = Z(23,5)$, $G = AF(23)$, $D_0 = \langle 0,1,2,4\rangle$, $D_1 = \langle 0,1,2,8\rangle$, $\mathcal{B} = \{\langle \emptyset, i+D_j, i+11+D_j\rangle \bmod 23, i \in I(11), j \in I(2)\}$.
23	9	36	$X = Z(23,5)$, $G = AF(23)$, $D_0 = \langle 0,1,3,4\rangle$, $D_1 = \langle 0,1,2,5\rangle$, $\mathcal{B} = \{\langle \emptyset, i+D_0, i+11+D_0\rangle \bmod 23, i \in I(11)\}$ (twice) $\cup \{\langle \emptyset, i+D_1, i+11+D_1\rangle \bmod 23, i \in I(11)\}$.
23	10	120	$X = Z(23,5)$, $G = AF(23)$, $D_0 = \langle 0,1,2,4,5\rangle$, $D_1 = \langle 0,2,5,8,10\rangle$, $D_2 = \langle 0,1,2,6,7\rangle$. $\mathcal{B} = \{\langle i+D_0, i+11+D_0\rangle \bmod 23, i \in I(11)\}$ (three times) $\cup \{\langle i+D_j, i+11+D_j\rangle \bmod 23, i \in I(11), j = 1,2\}$ (twice).
23	11	15	Unknown.
23	11	30	Unknown.
23	11	45	Lemma 3.6, $24 \in B_3(12,5)$—below.
23	11	60	$X = Z(23,5)$, $G = AF(23)$, $D_0 = \langle 2I(11)\rangle$, $D_1 = \langle 0,2,4,5,10\rangle$, $D_2 = \langle 0,1,2,4,8\rangle$, $\mathcal{B} = \{\langle i+D_0\rangle \bmod 23, i \in I(2)\}$ (three times) $\cup \{\langle \emptyset, i+D_j, i+11+D_j\rangle \bmod 23, i \in I(11), j = 1,2\}$.
23	11	75	$X = Z(23,5)$, $G = AF(23)$, $D_3 = \langle 0,1,2,4,5\rangle$, $D_4 = \langle 0,2,5,6,7\rangle$. $\mathcal{B} = \{\langle i+D_0\rangle \bmod 23, i \in I(2)\}$ $\cup \{\langle \emptyset, i+D_j, i+11+D_j\rangle \bmod 23, i \in I(11), j = 2,3,4\}$.
24	5	30	Lemma 3.5, $24 \in B_3(6,10)$—below, $G \in B_3(5,3)$—Lemma 3.1.
24	6	10	Denniston [5] shows that with $X = Z(23,5)$, and $\mathcal{B}$ the set of all distinct PSL(23)-images of $D_0 = \langle \infty, \emptyset, 0, 3, 16, 19\rangle$ then $(x, \mathcal{B})$ is a $B_3[6,10;24]$, since D_0 is stabilized by a subgroup of order 6, generated by the transformations $\alpha \mapsto \langle 19\rangle/\alpha$ and $\alpha \mapsto (\alpha + \langle 8\rangle)/(\alpha + \langle 5\rangle)$.
24	7	105	Lemma 3.5, $24 \in B_3(8,21)$—below, $8 \in B_3(7,5)$—Lemma 3.1.
24	8	21	Remark 3.9, $24 \in B_5(8,1)$ (see $21 \in B_3(7,60)$—above) and the same design proves $24 \in B_3(8,21)$.
24	9	84	Lemma 3.7, $23 \in B_3(8,24) \cap B_3(9,60)$—above.
24	10	180	Lemma 3.5, $24 \in B_3(12,5)$—below, $12 \in B_3(10,36)$—Lemma 3.1.
24	11	45	Lemma 3.5, $24 \in B_3(12,5)$—below, $12 \in B_3(11,9)$—Lemma 3.1.
24	12	5	Theorem 3.10.
25	5	3	Lemma 3.6, $26 \in B_3(6,1)$—below.
25	6	20	$X = GF(25, x^2 = 2x+2)$, $G = AF(25)$, $D_0 = \langle 0,1,5\rangle$, $D_1 = \langle 0,1,7\rangle$, $D_2 = \langle 0,2,3\rangle$, $D_3 = \langle 0,2,6\rangle$, $D_4 = \langle 4I(6)\rangle$; $\mathcal{B} = \{\langle i+D_j, i+12+D_j\rangle \bmod 25, i \in I(12), j \in I(3)\}$ (twice) $\cup \{\langle i+D_3, i+12+D_3\rangle \bmod 25, i \in I(12)\} \cup \{\langle i+D_4\rangle \bmod 24, i \in I(4)\}$ (twice).
25	7	35	$X = GF(25, x^2 = 2x+2)$, $G = AF(25)$, $D_5 = \langle 0,2,4\rangle$, $D_6 = \langle 0,3,4\rangle$, $D_7 = \langle 8I(3)\rangle$, $\mathcal{B} = \{\langle \emptyset, i+D_0, i+12+D_0\rangle \bmod 25, i \in I(12)\}$ (three times) $\cup \{\langle \emptyset, i+D_5, i+12+D_5\rangle \bmod 25, i \in I(12)\}$ (twice) $\cup \{\langle \emptyset, i+D_6, i+12+D_6\rangle \bmod 25, i \in I(12)\}$ $\cup \{\langle \emptyset, i+D_7, i+2j+1+D_7\rangle \bmod 25, i \in I(8), j \in I(2)\}$ $\cup \{\langle \emptyset, i+D_4\rangle \bmod 25, i \in I(4)\}$.
25	8	42	$X = GF(25, x^2 = 2x+2)$, $G = AF(25)$, $D_0 = \langle 0,1,4,8\rangle$, $D_1 = \langle 0,1,2,5\rangle$, $D_2 = \langle 6I(4)\rangle$, $D_3 = \langle 3I(8)\rangle$, $\mathcal{B} = \{\langle i+D_j, i+12+D_j\rangle \bmod 25, i \in (12), j \in I(2)\}$

Table 3.18. (Contd.)

v	k	λ	Construction
			$\cup\{\langle i+D_2, i+j+1+D_2\rangle \bmod 25, i\in(6), j\in I(2)\}$ (three times) $\cup\{\langle i+D_3\rangle \bmod 25, i\in I(3)\}$ (three times).
25	9	21	Unknown.
25	9	42	$X=\mathrm{GF}(25, x^2=2x+2)$, $G=\mathrm{AF}(25)$, $D_0=\langle 6I(4)\rangle$, $D_1=\langle 8I(3)\rangle$, $D_2=\langle 1,2,3,4\rangle$, $D_3=\langle 1,2,5,9\rangle$. $\mathcal{B}=\{\langle\emptyset, i+D_0, i+1+D_0\rangle \bmod 25, i\in I(6)\}$ $\cup\{\langle i+1+D_1, i+2+D_1, i+j+D_1\rangle \bmod 25, i\in I(8), j=5,6\}$ $\cup\{\langle\emptyset, i+D_j, i+12+D_j\rangle \bmod 25, i\in I(12), j=2,3\}$.
25	9	63	$X=\mathrm{GF}(25, x^2=2x+2)$, $G=\mathrm{AF}(25)$, $D_0=\langle 3I(8)\rangle$, $D_1=\langle 1,2,3,5\rangle$, $D_2=\langle 0,1,3,5\rangle$, $D_3=\langle 1,2,6,9\rangle$, $D_4=D_5=\langle 1,2,5,6\rangle$, $\mathcal{B}=\{\langle\emptyset, i+D_0\rangle \bmod 25, i\in I(3)\}$(three times) $\cup\{\langle\emptyset, i+D_j, i+12+D_j\rangle \bmod 25, i\in I(12), j\in N(5)\}$.
25	10	24	Unknown
25	10	48	$X=\mathrm{GF}(25, x^2=2x+2)$, $G=\mathrm{AF}(25)$, $D_0=\langle 1,3,4,5,8\rangle$, $D_1=\langle 1,2,5\rangle$, $D_2=\langle 1,2,3,4,8\rangle$, $D_3=\langle 1,2,3,10,11\rangle$. $\mathcal{B}=\{\langle\langle a\langle 1\rangle\oplus\langle D_0, 12+D_0\rangle\rangle+i\rangle, a\in \mathrm{GF}(5), i\in I(12)\}$ (twice) $\cup\{\langle\emptyset, i+D_1, i+8+D_1, i+16+D_1\rangle \bmod 25, i\in I(8)\}$ $\cup\{\langle i+D_j, i+12+D_j\rangle \bmod 25, i\in I(12), j=2,3\}$.
25	10	72	$X=\mathrm{GF}(25, x^2=2x+2)$, $G=\mathrm{AF}(25)$, $D_4=\langle 0,3,7,8,10,14,15,17,22,23\rangle$, $D_5=\langle 0,6,7,8,10,13,14,15,20,22\rangle$, $\mathcal{B}=\{\langle\langle a\langle 1\rangle\oplus\langle D_0, 12+D_0\rangle\rangle+i\rangle, a\in \mathrm{GF}(5), i\in I(12)\}$ (three times) $\cup\{\langle i+D_j\rangle \bmod 25, i\in I(24), j=4,5\}$.
25	11	495	Lemma 3.7, $24\in B_3(10,180)\cap B_3(11,45)$—above.
25	12	110	Lemma 3.6, $26\in B_3(13,11)$—below.
26	5	1	$X=Z(2,1)\times Z(13,2)$, $G=\mathrm{AF}(2)\times \mathrm{AF}(13)$, $D_0=\langle 0,6\rangle$, $D_1=\langle 4I(3)\rangle$, $\mathcal{B}=\{\langle(\emptyset, i+D_0), (0,\emptyset), (0, i+2+D_0)\rangle \bmod(2, 13), i\in I(6)\}$ $\cup\{\langle(\emptyset,\emptyset), (0,\emptyset), (0, i+D_1)\rangle \bmod(2, 13), i\in I(4)\}$.
26	6	1	Theorem 3.12.
26	7	35	Theorem 3.13.
28	8	7	Unknown.
26	8	14	$X=\mathrm{GF}(25, x^2=2x+2)\cup\{\infty\}$, $G=\mathrm{AF}(25)$, $D_0=\langle 4I(6)\rangle$, $D_1=\langle 6I(4)\rangle$, $\mathcal{B}=\{\langle\infty, \emptyset, i+D_0\rangle \bmod 25, i\in I(4)\}$ (twice) $\cup\{\langle i+D_1, i+1+D_1\rangle \bmod 25, i\in I(6)\}$ $\cup\{\langle i+D_1, i+2+D_1\rangle \bmod 25, i\in I(6)\}$ (twice).
26	8	21	Lemma 3.15.
26	9	21	Unknown.
26	9	42	$X=\mathrm{GF}(25, x^2=2x+2)\cup\{\infty\}$, $G=\mathrm{AF}(25)$, $D_0\langle 1,2,4,6\rangle$, $D_1=\langle 3I(8)\rangle$, $D_2=\langle 6I(4)\rangle$, $D_3=\langle 8I(3)\rangle$, $D_4=\langle 1,2,6,11\rangle$, $\mathcal{B}=\{\langle\infty, i+D_0, i+12+D_0\rangle \bmod 25, i\in I(12)\}$ $\cup\{\langle\infty, i+D_1\rangle \bmod 25, i\in I(3)\}$ (twice) $\cup\{\langle\emptyset, i+D_2, i+1+D_2\rangle \bmod 25, i\in I(6)\}$ $\cup\{\langle i+D_3, i+1+D_3, i+j+D_3\rangle \bmod 25, i\in I(8), j=2,5\}$ $\cup\{\langle\emptyset, i+D_4, i+12+D_4\rangle \bmod 25, i\in I(12)\}$.
26	9	63	Theorem 3.13.
26	10	3	Does not exist by Theorem 2.4.
26	10	$3i$	$2\leqslant i\leqslant 7$ unknown.
26	10	24	$X=\mathrm{GF}(25, x^2=2x+2)\cup\{\infty\}$, $G=\mathrm{AF}(25)$,

Table 3.18. (Contd.)

v	k	λ	Construction
			$D_0=\langle 8I(3)\rangle$, $D_1=\langle 1,3,4,5,8\rangle$,
			$\mathcal{B}=\{\langle\infty, i+D_0, i+1+D_0, i+5+D_0\rangle \bmod 25, i\in I(8)\}$
			$\cup\{\langle\langle a\langle 1\rangle\oplus\langle D_1, 12+D_1\rangle\rangle+i\rangle, a\in \mathrm{GF}(5), i\in I(12)\}$ (twice)
			$\cup\{\langle\emptyset, i+D_0, i+1+D_0, i+4+D_0\rangle \bmod 25, i\in I(8)\}$.
26	10	$3i$	$i=9, 10, 11$ unknown.
26	10	36	$X=\mathrm{GF}(25, x^2=2x+2)\cup\{\infty\}$, $G=\mathrm{AF}(25)$,
			$D_2=\langle 1,2,9,11\rangle$, $D_3=\langle 1,2,3,7,9\rangle$,
			$\mathcal{B}=\{\langle\infty, \emptyset, i+D_2, i+12+D_2\rangle \bmod 25, i\in I(12)\}$
			$\cup\{\langle\langle a\langle 1\rangle\oplus\langle D_1, 12+D_1\rangle\rangle+i\rangle, a\in \mathrm{GF}(5), i\in I(12)\}$ (three times)
			$\cup\{\langle i+D_3, i+12+D_3\rangle \bmod 25, i\in I(12)\}$.
26	10	$3i$	$i=13, 14$ unknown.
26	10	45	Theorem 3.12.
26	10	$3i$	$i>15$, see Section 4 below.
26	11	33	Unknown.
26	11	66	$X=\mathrm{GF}(25, x^2=2x+2)\cup\{\infty\}$, $G=\mathrm{AF}(25)$,
			$D_0=\langle 1,3,4,5,8\rangle$, $D_1=\langle 8I(3)\rangle$, $D_2=\langle 0,1,8,9,11\rangle$,
			$D_3=\langle 0,2,5,9,11\rangle$,
			$\mathcal{B}=\{\infty, \langle\langle a\langle 1\rangle\oplus\langle D_0, 12+D_0\rangle\rangle+i\rangle, a\in \mathrm{GF}(5), i\in I(12)\}$ (four times)
			$\cup\{\infty, \langle\emptyset, i+D_1, i+1+D_1, i+5+D_1\rangle \bmod 25, i\in I(8)\}$
			$\cup\{\langle\emptyset, i+D_j, i+12+D_j\rangle \bmod 25, i\in I(12), j=2,3\}$.
26	11	99	$X=\mathrm{GF}(25, x^2=2x+2)\cup\{\infty\}$, $G=\mathrm{AF}(25)$,
			$D_4=\langle 0,3,7,8,10,14,15,17,22,23\rangle$,
			$D_5=\langle 1,2,3,6,11\rangle$, $D_6=D_7=D_2$,
			$\mathcal{B}=\{\infty, \langle\langle a\langle 1\rangle\oplus\langle D_0, 12+D_0\rangle\rangle+i\rangle, a\in \mathrm{GF}(5), i\in I(12)\}$
			$\cup\{\langle\infty, i+D_4\rangle \bmod 25, i\in I(24)\}$
			$\cup\{\langle\emptyset, i+D_j, i+12+D_j\rangle \bmod 25, i\in I(12), j=5,6,7\}$.
26	12	55	Lemma 3.15.
26	13	11	Theorem 3.10.
27	5	6	$X=\mathrm{GF}(27, x^3=x+2)$, $G=\mathrm{AF}(27)$,
			$D_0=\langle 0,1,2,20\rangle$, $D_1=\langle 0,2,3,7\rangle$, $D_2=\langle 0,1\rangle$.
			$\mathcal{B}=\{\langle\emptyset, i+D_j\rangle \bmod 27, i\in I(26), j\in I(2)\}$
			$\cup\{\langle\emptyset, i+D_2, i+12+D_2\rangle \bmod 27, i\in I(13)\}$.
27	6	4	$X=\mathrm{GF}(27, x^3=x+2)$, $G=\mathrm{AF}(27)$,
			$D_0=\langle 1,3,9\rangle$, $D_1=\langle 1,2,5\rangle$.
			$\mathcal{B}=\{\langle\langle a\langle 2\rangle\oplus b\langle 1\rangle\oplus\langle D_0, 13+D_0\rangle\rangle+i\rangle, a, b\in I(3), i\in I(13)\}$ (twice)
			$\cup\{\langle i+D_1, i+13+D_1\rangle \bmod 27, i\in I(13)\}$.
27	7	21	$X=\mathrm{GF}(27, x^3=x+2)$, $G=\mathrm{AF}(27)$,
			$D_j=\langle 0,1,3,6,10,8j+16\rangle$, $j\in I(2)$, $D_2=\langle I(3)\rangle$,
			$\mathcal{B}=\{\langle\emptyset, i+D_j\rangle \bmod 27, i\in I(26), j\in I(2)\}$
			$\cup\{\langle\emptyset, i+D_2, i+13+D_2\rangle \bmod 27, i\in I(13)\}$.
27	8	168	Lemma 3.6, $28\in B_3(9,28)$—below.
27	9	28	$X=\mathrm{GF}(27, x^3=x+2)$, $G=\mathrm{AF}(27)$,
			$D_0=\langle 0,1,3,9\rangle$, $D_1=\langle 0,1,3,6,9,10,11,13\rangle$,
			$D_2=\langle 0,1,3,4,7,9,13,18\rangle$, $D_3=\langle 0,2,5,12,13,17,20,21\rangle$,
			$D_4=\langle 0,1,2,3,9,12,13,21\rangle$,
			$\mathcal{B}=\{\langle\langle a\langle 2\rangle\oplus\langle\emptyset, D_0, 13+D_0\rangle\rangle+i\rangle, a\in I(3), i\in I(13)\}$
			$\cup\{\langle\langle a\langle 2\rangle\oplus b\langle 1\rangle\oplus\langle\emptyset, D_j\rangle\rangle+i\rangle, a, b\in I(3), i\in I(26), j\in N(4)\}$.
27	10	72	$X=\mathrm{GF}(27, x^3=x+2)$, $G=\mathrm{AF}(27)$,
			$D_0=D_1=\langle 1,2,3,5,6\rangle$, $D_2=D_3=\langle 1,2,4,5,8\rangle$,
			$D_4=\langle 1,2,4,9,10\rangle$,
			$\mathcal{B}=\{\langle i+D_j, i+13+D_j\rangle \bmod 27, i\in I(13), j\in I(5)\}$.
27	11	99	$X=\mathrm{GF}(27, x^3=x+2)$, $G=\mathrm{AF}(27)$,

Table 3.18. (Contd.)

v	k	λ	Construction
			$D_0=\langle 0,2,5,8,9\rangle$, $D_1=\langle 0,2,3,5,7,8,12,13,17,23,25\rangle$, $D_2=\langle 0,2,3,4,7,8,12,13,17,19,21\rangle$. $\mathscr{B}=\{\langle\emptyset, i+D_0, i+13+D_0\rangle \bmod 27, i\in I(13)\}\cup\{\langle i+D_j\rangle \bmod 27, i\in I(26), j=1,2\}$.
27	12	44	$X=\mathrm{GF}(27, x^3=x+2)$, $G=\mathrm{AF}(27)$, $D_0=\langle 0,1,2,3,9,12,13,14,16,21,22\rangle$, $D_1=\langle 0,1,2,3,6,10\rangle$, $\mathscr{B}=\{\langle\langle a\langle 2\rangle\oplus b\langle 1\rangle\oplus\langle\emptyset, D_0\rangle\rangle+i\rangle, a,b\in I(3), i\in I(26)\}$ $\cup\{\langle i+D_1, i+13+D_1\rangle \bmod 27, i\in I(13)\}$.
27	13	66	Lemma 3.6, $28\in B_3(14,6)$—below.
28	5	30	Lemma 3.5, $28\in B_3(6,10)$—below, $6\in B_3(5,3)$—Lemma 3.1.
28	6	10	$X=\mathrm{GF}(27, x^3=x+2)\cup\{\infty\}$, $G=\mathrm{AF}(27)$, $D_0=\langle 1,2\rangle$, $D_1=\langle 1,3,9\rangle$, $D_2=\langle 1,2,7\rangle$, $D_3=\langle 1,2,8\rangle$, $D_4=\langle 1,3,5\rangle$, $\mathscr{B}=\{\langle\infty,\emptyset, i+D_0, i+13+D_0\rangle \bmod 27, i\in I(13)\}$ $\cup\{\langle\langle a\langle 2\rangle\oplus b\langle 1\rangle\oplus\langle D_1, 13+D_1\rangle\rangle+i\rangle, a,b\in I(3), i\in I(13)$ (twice) $\cup\{\langle i+D_j, i+13+D_j\rangle \bmod 27, i\in I(13), j\in 2,3,4)$.
28	7	5	Unknown.
28	7	10	$X=\mathrm{GF}(27, x^3=x+2)\cup\{\infty\}$, $G=\mathrm{AF}(27)$, $D_0=\langle 0,1,3\rangle$, $D_1=\langle 1,2,10\rangle$, $D_2=\langle 0,1,3,6,10,24\rangle$. $\mathscr{B}=\{\infty,\langle\langle a\langle 2+8j\rangle\oplus b\langle 3-3j\rangle\oplus\langle D_j, 13+D_j\rangle\rangle+i\rangle, a,b\in I(3), i\in I(13), j\in I(2)\}$ $\cup\{\langle\emptyset, i+D_2\rangle \bmod 27, i\in I(26)\}$.
28	7	15	Denniston [5] shows that with $X=\mathrm{GF}(27, x^3=x+2)\cup\{\infty\}$ and B, the set of all distinct PSL(27)-images of $D_0=\langle\infty,\emptyset,0,4,7,15,23\rangle$ then (X,B) is a $B_3[7,15,28]$, since D_0 is stabilized by a subgroup of order 7 (generated by the transformation $\alpha\to\langle 17\rangle/(\langle 17\rangle\oplus\alpha)$.
28	8	42	$X=\mathrm{GF}(27, x^3=x+2)\cup\{\infty\}$, $G=\mathrm{AF}(27)$, $D_0=\langle 0,1,3,6,10,22\rangle$, $D_1=\langle 0,1,2,3,6,10,23\rangle$, $D_2=\langle 0,1,3,6,10,12,19\rangle$, $D_3=\langle I(4)\rangle$, $\mathscr{B}=\{\langle\infty,\emptyset, i+D_0\rangle \bmod 27, i\in I(26)\}$ $\cup\{\langle\emptyset, i+D_j\rangle \bmod 27, i\in I(26), j\in N(2)\}$ $\cup\{\langle i+D_3, i+13+D_3\rangle \bmod 27, i\in I(13)\}$.
28	9	28	$X=\mathrm{GF}(27, x^3=x+2)\cup\{\infty\}$, $G=\mathrm{PGL}(27)$. Let $D=\{0,1,2,x,x+1,x+2,2x,2x+1,2x+2\}$. Then D is stabilized by a subgroup of order 18 generated by: $\alpha\to\alpha+1$; $\alpha\to\alpha+x$; and $\alpha\to-\alpha$. Since G is triply transitive, the $\lvert G\rvert/18$ distinct images of D produce our design.
28	10	20	Unknown.
28	10	40	$X=\mathrm{GF}(27, x^3=x+2)\cup\{\infty\}$, $G=\mathrm{AF}(27)$, $D_0=\langle 0,1,3,9\rangle$, $D_1=\langle 0,1,3,6,9,10,11,13\rangle$, $D_2=\langle 1,2,3,4,8\rangle$, $D_3=\langle 1,2,3,5,10\rangle$, $\mathscr{B}=\{\infty,\langle\langle a\langle 2\rangle\oplus\langle\emptyset, D_0\rangle\rangle+i\rangle, a\in I(3), i\in I(13)\}$ (four times) $\cup\{\infty,\langle\langle a\langle 2\rangle\oplus b\langle 1\rangle\oplus\langle\emptyset, D_1\rangle\rangle+i\rangle, a,b\in I(3), i\in I(26)\}$ $\cup\{\langle i+D_j, i+13+D_j\rangle \bmod 27, i\in I(13), j=2,3\}$.
28	10	60	$X=\mathrm{GF}(27, x^3=x+2)\cup\{\infty\}$, $G=\mathrm{AF}(27)$, $D_4=\langle 0,1,3,4,7,9,13,18\rangle$, $D_5=\langle 1,2,4,6,8\rangle$, $\mathscr{B}=\{\infty,\langle\langle a\langle 2\rangle\oplus\langle\emptyset, D_0, 13+D_0\rangle\rangle+i\rangle, a\in I(3), i\in I(13)\}$ (three times) $\cup\{\infty,\langle\langle a\langle 2\rangle\oplus b\langle 1\rangle\oplus\langle\emptyset, D_j\rangle\rangle+i\rangle, a,b\in I(3), i\in I(3), i\in I(26), j=1,4\}$ $\cup\{\langle i+D_5, i+13+D_5\rangle \bmod 27, i\in I(13)\}$ (three times).
28	11	495	Lemma 3.5, $28\in B_3(12,55)$—below, $12\in B_3(11,9)$—Lemma 3.1.
28	12	55	$X=Z(2,1)\times Z(13,2)\cup\{\infty_0,\infty_1\}$, $G=\mathrm{AF}(2)\times(\mathrm{AF}(13)/\langle\alpha\to\langle 4\rangle\alpha\rangle)$, $D_i=\langle i+4I(3)\rangle$, $i\in I(3)$, $D_3=\langle 2I(6)\rangle$, $D_4=\langle 0,4,6,10\rangle$, $D_5=\langle 3I(4)\rangle$, $D_6=\langle 0,1,6,7\rangle$, $D_7=\langle 0,1,3,6,7,9\rangle$, $D_8=\langle 0,1,2,5,6,7,8,11\rangle$,

Table 3.18. (Contd.)

v	k	λ	Construction
			$\mathscr{B}=\{\langle\infty_0,\infty_1,(\emptyset,i+D_1),(0,\emptyset),(0,i+D_3)\rangle \bmod(2,13), i\in I(4)\}$ $\cup\{\langle\infty_0,\infty_1,(\emptyset,\emptyset),(\emptyset,i+D_5),(0,\emptyset),(0,i+D_5)\rangle \bmod(-,13), i\in I(3)\}$ $\cup\{\langle\infty_0,(\emptyset,\emptyset),(\emptyset,i+D_4),(0,i+D_7)\rangle \bmod(2,13), i\in I(6)\}$ $\cup\{\langle\infty_1,(\emptyset,\emptyset),(\emptyset,i+D_6),(0,i+D_3)\rangle \bmod(2,13), i\in I(6)\}$ $\cup\{\langle\infty_i,(\emptyset,\emptyset),(\emptyset,j+D_{2i}),(0,\emptyset),(0,j+D_3)\rangle \bmod(2,13), i\in I(2), j\in I(2)\}$ $\cup\{\langle(\emptyset,i+D_5),(0,i+D_8)\rangle \bmod(2,13), i\in I(6)\}$ $\cup\{\langle(\emptyset,i+D_0),(\emptyset,i+D_1),(0,i+D_3)\rangle \bmod(2,13), i\in I(4)\}$.
28	13	22	Unknown.
28	13	44	Unknown.
28	13	66	Lemma 3.15.
28	13	88	Unknown.
28	13	110	Lemma 3.7, $27\in B_3(12,44)\cap B_3(13,66)$—above.
28	13	154	Unknown.
28	14	6	Theorem 3.10.
29	5	5	$X=Z(29,2)$, $G=AF(29)$, $D_0=\langle 0,1,6,9\rangle$, $D_1=\langle 0,1,6,24\rangle$, $D_2=\langle 7I(4)\rangle$, $\mathscr{B}=\{\langle\emptyset,i+D_j\rangle \bmod 29, i\in I(28), j\in I(2)\}\cup\{\langle\emptyset,i+D_2\rangle \bmod 27, i\in I(7)\}$.
29	6	20	$X=Z(29,2)$, $G=AF(29)$, $D_i=\langle 0,1,8+i\rangle$, $i\in I(2)$, $D_{2+i}=\langle 0,1,3+i\rangle$, $i\in I(2)$, $D_4=\langle 0,1,5\rangle$, $D_5=\langle 0,1,10\rangle$, $\mathscr{B}=\{\langle i+D_j,i+14+D_j\rangle \bmod 29, i\in I(14), j=0,1,4\}$ (twice) $\cup\{\langle i+D_j,i+14+D_j\rangle \bmod 29, i\in I(14), j=2,3,5\}$.
29	7	5	Unknown.
29	7	10	$X=Z(29,2)$, $G=AF(29)$, $D_0=\langle 4I(7)\rangle$, $D_1=\langle 0,5,6,10,12,18,24\rangle$, $\mathscr{B}=\{\langle i+D_0\rangle \bmod 29, i\in I(4)\}$ (twice) $\cup\{\langle i+D_1\rangle \bmod 29, i\in I(28)\}$.
29	7	15	$X=Z(29,2)$, $G=AF(29)$, $D_2=\langle 0,6,10\rangle$, $D_3=\langle 0,9,14,17,19,22\rangle$, $\mathscr{B}=\{\langle i+D_0\rangle \bmod 29, i\in I(4)\}$ (three times) $\cup\{\langle\emptyset,i+D_2,i+14+D_2\rangle \bmod 29, i\in I(14)\}$ $\cup\{\langle\emptyset,i+D_3\rangle \bmod 29, i\in I(28)\}$.
29	8	4	Unknown.
29	8	8	$X=Z(29,2)$, $G=AF(29)$, $D_0=\langle 4I(7)\rangle$, $D_1=\langle 0,1,2,10\rangle$, $\mathscr{B}=\{\langle\emptyset,i+D_0\rangle \bmod 29, i\in I(4)\}\cup\{\langle i+D_1,i+14+D_1\rangle \bmod 29, i\in I(14)\}$.
29	8	12, 20	Unknown.
29	8	16,24	Lemma 3.4.
29	8	28	$X=Z(29,2)$, $G=AF(29)$, $D_0=\langle 7I(4)\rangle$, $D_1=\langle 0,5,6,10,12,13,18,25\rangle$, $D_2=\langle 0,4,5,6,10,12,23,25\rangle$, $\mathscr{B}=\{\langle i+D_0,i+2+D_0\rangle \bmod 29, i\in I(7)\}\cup\{\langle i+D_j\rangle \bmod 29, i\in I(28), j=1,2\}$.
29	9	14	Unknown.
29	9	28	$X=Z(29,2)$, $G=AF(29)$, $D_0=\langle 0,1,3,12\rangle$, $D_1=\langle 0,1,6,8\rangle$, $D_2=\langle 0,1,7,9\rangle$, $\mathscr{B}=\{\langle\emptyset,i+D_j,i+14+D_j\rangle \bmod 29, i\in I(14), j\in I(3)\}$.
29	9	42	$X=Z(29,2)$, $G=AF(29)$, $D_0=\langle 0,1.3,10\rangle$, $D_1=\langle 0,1,4,11\rangle$, $D_2=\langle 0,1,5,11\rangle$, $D_3=\langle 0,1,6,12\rangle$, $D_4=\langle 7I(4)\rangle$, $\mathscr{B}=\{\langle\emptyset,i+D_j,i+14+D_j\rangle \bmod 29, i\in I(14), j\in I(4)\}$ $\cup\{\langle\emptyset,i+D_4,i+2+D_4\rangle \bmod 29, i\in I(7)\}$.
29	10	40	$X=Z(29,2)$, $G=AF(29)$, $D_i=\langle 0,1,2,3+i,7\rangle$, $i\in I(2)$, $D_2=\langle 0,1,2,4,10\rangle$,

Table 3.18. (Contd.)

v	k	λ	Construction
			$\mathcal{B}=\{\langle i+D_j, i+14+D_j\rangle \bmod 29, i\in I(14), j\in I(3)\}$.
29	11	55	$X=Z(29,2)$, $G=\mathrm{AF}(29)$, $D_i=\langle 0,1,3,4+i,12-i\rangle$, $i\in I(2)$, $D_2=\langle 0,1,2,8,9\rangle$, $\mathcal{B}=\{\langle\emptyset, i+D_j, i+14+D_j\rangle \bmod 29, i\in I(14), j\in I(3)\}$.
29	12	110	$X=Z(29,2)$, $G=\mathrm{AF}(29)$, $D_i=\langle 0,1,2,3,6+i,9+i\rangle$, $i\in I(2)$, $D_{2+i}=\langle 0,1,2,3,4+i\ 6+5i\rangle$, $i\in I(2)$, $D_4=\langle 7I(4)\rangle$, $\mathcal{B}=\{\langle i+D_j, i+14+D_j\rangle \bmod 29, i\in I(14), j\in I(4)\}$ $\cup\{\langle i+D_4, i+1+D_4, i+4+D_4\rangle \bmod 29, i\in I(7)\}$.
29	13	143	$X=Z(29,2)$, $G=\mathrm{AF}(29)$, $D_i=\langle 0,1,2,4-i,7+i,8+2i\rangle$, $i\in I(2)$, $D_2=\langle 0,1,2,3,9,12\rangle$, $D_3=\langle 7I(4)\rangle$, $\mathcal{B}=\{\langle\emptyset, i+D_j, i+14+D_j\rangle \bmod 29, i\in I(14), j\in I(2)\}$ $\cup\{\langle\emptyset, i+D_2, i+14+D_2\rangle \bmod 29, i\in I(14)\}$ (twice) $\cup\{\langle\emptyset, i+D_3, i+2+D_3\ i+4+D_3\rangle \bmod 29, i\in I(7)\}$.
29	14	52	Unknown.
29	14	104	$X=Z(29,2)$, $G=\mathrm{AF}(29)$, $D_0=\langle 2I(14)\rangle$, $D_1=\langle 0,1,2,5,6,9,12\rangle$, $D_2=\langle 0,1,2,5,7,11,12\rangle$, $\mathcal{B}=\{\langle i+D_0\rangle \bmod 29, i\in I(2)\}$ (four times) $\cup\{\langle i+D_j, i+14+D_j\rangle \bmod 29, i\in I(14), j=1,2\}$.
29	14	156	Lemma 3.6, $30\in B_3(15,13)$—below.
30	5	3	$X=Z(29,2)\cup\{\infty\}$, $G=\mathrm{SAF}(29)$, $D_0=\langle 7I(4)\rangle$, $D_1=\langle 0,1,3,21\rangle$, $D_2=\langle 1,3,5,11\rangle$, $D_3=\langle 0,5,14,19\rangle$, $\mathcal{B}=\{\langle\infty, i+D_0\rangle \bmod 29, i\in I(7)\}\cup\{\langle\emptyset, 2i+D_3\rangle \bmod 29, i\in I(7)\}$ $\cup\{\langle\emptyset, 2i+D_{j+1}\rangle \bmod 29, i\in I(14), j\in I(2)\}$.
30	6	5	Theorem 3.12.
30	7	15	Lemma 3.15.
30	8	$6i$	$1\leqslant i\leqslant 6$ unknown. In several of the designs with parameters $v=30$, $k=8$ we shall be using $X=Z(29,2)\cup\{\infty\}$ and the group $G=\mathrm{AF}(29)$. To streamline our presentation we shall list the various D_i and the corresponding sets $\mathcal{B}_i$ that we need. $D_0=\langle 4I(7)\rangle$, $\mathcal{B}_0=\{\langle\infty, i+D_0\rangle \bmod 29, i\in I(4)\}$. $D_1=\langle 4I(7)\rangle$, $\mathcal{B}_1=\{\langle\emptyset, i+D_0\rangle \bmod 29, i\in I(4)\}$. $D_2=\langle 7I(4)\rangle$, $\mathcal{B}_2=\{\langle i+D_2, i+3+D_2\rangle \bmod 29, i\in I(7)\}$. $D_3=\langle 0,5,8,9\rangle$, $D_4=\langle 0,1,2,7\rangle$, $\mathcal{B}_j=\{\langle i+D_j, i+14+D_j\rangle \bmod 29$, $i\in I(14)$, $j=3,4\}$. $D_5=\langle 0,5,6,7,10,12,14\rangle$, $D_6=\langle 0,5,6,7,10,12,23\rangle$, $\mathcal{B}_j=\{\langle\infty, i+D_j\rangle \bmod 29, i\in I(28), j=5,6\}$. $D_7=\langle 0,5,6,10,12,13,18,25\rangle$, $D_8=\langle 0,5,6,10,12,17,18,23\rangle$, $D_9=\langle 0,5,6,10,12,13,23,26\rangle$, $D_{10}=\langle 0,4,9,14,16,17,21,26\rangle$, $D_{11}=\langle 0,5,6,7,10,11,12,25\rangle$, $D_{12}=\langle 0,1,2,7,14,22,25,26\rangle$, $D_{13}=\langle 0,2,4,6,9,10,16,21\rangle$, $\mathcal{B}_j=\{\langle i+D_j\rangle \bmod 29, i\in I(28), 7\leqslant j\leqslant 13\}$.
30	8	42	$\mathcal{B}=\{\mathcal{B}_i: i=2,3,5,7,8\}$.
30	8	48	Theorem 3.13.
30	8	54	Lemma 3.7, $29\in B_3(7,10)\cap B_3(8,44)$—above.
30	8	60	$\mathcal{B}=\mathcal{B}_0$(three times)$\cup\mathcal{B}_1$(three times)$\cup\mathcal{B}_4\cup\mathcal{B}_6\cup\mathcal{B}_7$(twice)$\cup\mathcal{B}_{11}$.
30	8	66	$\mathcal{B}=\mathcal{B}_0$(four times)$\cup\mathcal{B}_1$(four times)$\cup\{\mathcal{B}_i: i=2,4,6,8,10,11\}$.
30	8	72	$\mathcal{B}=\mathcal{B}_0$(five times)$\cup\mathcal{B}_1$(five times)$\cup\mathcal{B}_6\cup\mathcal{B}_7\cup\mathcal{B}_8\cup\mathcal{B}_{12}$(twice).
30	8	78	$\mathcal{B}=\mathcal{B}_0$(six times)$\cup\mathcal{B}_1$(six times)$\cup\mathcal{B}_2\cup\mathcal{B}_6\cup\mathcal{B}_9$(twice)$\cup\mathcal{B}_{10}\cup\mathcal{B}_{13}$.
30	9	$6i$	$1\leqslant i\leqslant 4$ unknown.

Table 3.18. (Contd.)

v	k	λ	Construction
30	9	30	$X=Z(29,2)\cup\{\infty\}$, $G=\mathrm{AF}(29)$, $D_0=\langle 4I(7)\rangle$, $D_1=\langle 7I(4)\rangle$, $D_2=\langle 0,5,7,8,10,12,15,17,21\rangle$. $\mathcal{B}=\{\langle\infty,\emptyset,i+D_0\rangle \bmod 29, i\in I(4)\}$ (twice) $\cup\{\langle\infty,i+D_1,i+1+D_1\rangle \bmod 29, i\in I(7)\}$ $\cup\{\langle\emptyset,i+D_1,i+2+D_1\rangle \bmod 29, i\in I(7)\}\cup\{\langle i+D_2\rangle \bmod 29, i\in I(28)\}$.
30	9	36	Lemma 3.7, $29\in B_3(8,8)\cap B_3(9,28)$—above.
30	9	42	$X=Z(29,2)\cup\{\infty\}$, $G=\mathrm{AF}(29)$, $D_3=\langle 0,1,5,11\rangle$, $D_4=\langle 0,2,5,11\rangle$, $D_5=\{0,5,7,10,12,17,19,20,21\rangle$, $\mathcal{B}=\{\langle\infty,i+D_1,i+2+D_1\rangle \bmod 29, i\in I(7)\}$ $\cup\{\langle\infty,i+D_3,i+14+D_3\rangle \bmod 29, i\in I(14)\}$ $\cup\{\langle\emptyset,i+D_1,i+3+D_1\rangle \bmod 29, i\in I(7)\}$ $\cup\{\langle\emptyset,i+D_4,i+14+D_4\rangle \bmod 29, i\in I(14)\}\cup\{\langle i+D_5\rangle \bmod 29, i\in I(28)\}$.
30	9	48	$X=Z(29,2)\cup\{\infty\}$, $G=\mathrm{AF}(29)$, $D_6=\langle 0,2,5,7\rangle$, $D_7=\langle 0,1,5,10\rangle$, $D_8=\langle 0,1,2,7,8,13,18,22,25\rangle$, $\mathcal{B}=\{\langle\infty,\emptyset,i+D_0\rangle \bmod 29, i\in I(4)\}$(six times) $\cup\{\langle\emptyset,i+D_j,i+14+D_j\rangle \bmod 29, i\in I(14), j=6,7\}$ $\cup\{\langle i+D_8\rangle \bmod 29, i\in I(28)\}$.
30	9	54	$X=Z(29,2)\cup\{\infty\}$, $G=\mathrm{AF}(29)$, $D_9=\langle 0,5,7,10,12,15,17,20,21\rangle$, $D_{10}=\langle 0,5,7,8,10,12,16,21,26\rangle$. $\mathcal{B}=\{\langle\infty,\emptyset,i+D_0\rangle \bmod 29, i\in I(4)\}$ (five times) $\cup\{\langle\infty,i+D_1,i+2+D_1\rangle \bmod 29, i\in I(7)\}$ $\cup\{\langle\infty,i+D_1,i+2+D_1\rangle \bmod 29, i\in I(7)\}$ $\cup\{\langle i+D_j\rangle \bmod 29, i\in I(28), j=9,10\}$.
30	10	18	$X=Z(29,2)\cup\{\infty\}$, $G=\mathrm{AF}(29)$, $D_0=\langle 7I(4)\rangle$, $D_1=\langle 0,1,2,10,11\rangle$, $\mathcal{B}=\{\langle\infty,\emptyset,i+D_0,i+1+D_0\rangle \bmod 29, i\in I(7)\}$ $\cup\{\langle i+D_1,i+14+D_1\rangle \bmod 29, i\in I(14)\}$.
30	11	495	Lemma 3.5, $30\in B_3(12,55)$—below, $12\in B_3(11,9)$—Lemma 3.1.
30	12	55	$X=Z(29,2)\cup\{\infty\}$, $G=\mathrm{AF}(29)$, $D_0=\langle 0,1,3,5,7\rangle$, $D_1=\langle 0,1,2,3,5,8\rangle$, $D_2=\langle 7I(4)\rangle$, $\mathcal{B}=\{\langle\infty,\emptyset,i+D_0,i+14+D_0\rangle \bmod 29, i\in I(14)\}$ $\cup\{\langle i+D_1,i+14+D_1\rangle \bmod 29, i\in I(14)\}$ $\cup\{\langle i+D_2,i+1+D_2,i+2+D_2\rangle \bmod 29, i\in I(7)\}$.
30	13	429	$X=Z(29,2)\cup\{\infty\}$, $G=\mathrm{AF}(29)$, $D_0=\langle 0,1,2,3,4,6\rangle$, $D_1=\langle 0,1,2,3,8,11\rangle$, $D_2=\langle 7I(4)\rangle$, $D_3=\langle 0,1,2,3,5,12\rangle$, $D_4=\langle 0,1,2,3,8,10\rangle$, $D_5=\langle 0,1,2,6,11,12\rangle$, $\mathcal{B}=\{\infty,i+D_0,i+14+D_0\rangle \bmod 29, i\in I(14)\}$ (five times) $\cup\{\langle\infty,i+D_1,i+14+D_1\rangle \bmod 29, i\in I(14)\}$ $\cup\{\langle\infty,i+D_2,i+2+D_2,i+4+D_2\rangle \bmod 29, i\in I(7)\}$ $\cup\{\langle\emptyset,i+D_3,i+14+D_3\rangle \bmod 29, i\in I(14)\}$ (four times) $\cup\{\langle\emptyset,i+D_4,i+14+D_4\rangle \bmod 29, i\in I(14)\}$ (three times) $\cup\{\langle\emptyset,i+D_5,i+14+D_5\rangle \bmod 29, i\in I(14)\}$ $\cup\{\langle\emptyset,i+D_2,i+2+D_2,i+4+D_2\rangle \bmod 29, i\in I(7)\}$.
30	14	39	Unknown.
30	14	78	Lemma 3.15.
30	14	117	$X=Z(29,2)\cup\{\infty\}$, $G=\mathrm{AF}(29)$, $D_0=\langle 7I(4)\rangle$, $D_1=\langle 0,1,2,3,5,11\rangle$, $D_2=\langle 2I(14)\rangle$, $D_3=\langle 0,1,2,5,6,9,12\rangle$, $\mathcal{B}=\{\langle\infty,\emptyset,i+D_0,i+1+D_0,i+4+D_0\rangle \bmod 29, i\in I(7)\}$ $\cup\{\langle\infty,\emptyset,i+D_1,i+14+D_1\rangle \bmod 29, i\in I(14)\}$ $\cup\{\langle i+D_2\rangle \bmod 29, i\in I(2)\}$ (five times)

Table 3.18. (Contd.)

v	k	λ	Construction
			$\cup\{\langle i+D_3, i+14+D_3\rangle \bmod 29, i\in I(14)\}$.
30	15	13	Theorem 3.10.
31	5	6	$X=Z(31,3)$, $G=\mathrm{AF}(31)$,
			$D_j=\langle 0,1,2,13+13i\rangle$, $i\in I(2)$, $D_2=\langle 0,3\rangle$, $D_3=\langle 6I(5)\rangle$,
			$\mathscr{B}=\{\langle\emptyset, i+D_j\rangle \bmod 31, i\in I(30), j\in I(2)\}\cup\{\langle\emptyset, i+D_2, i+15+D_2\rangle \bmod 31, i\in I(15)\}$.
			$\cup\{\langle i+D_3\rangle \bmod 31, i\in I(6)\}$ (twice).
31	6	4	Unknown.
31	6	8	$X=Z(31,3)$, $G=\mathrm{AF}(31)$,
			$D_0=\langle 0,2,9\rangle$, $D_1=\langle 0,3,10\rangle$,
			$D_2=\langle 6I(5)\rangle$, $D_3=\langle 5I(6)\rangle$,
			$\mathscr{B}=\{\langle i+D_j, i+15+D_j\rangle \bmod 31, i\in I(15), j\in I(2)\}$
			$\cup\{\langle\emptyset, i+D_2\rangle \bmod 31, i\in I(6)\}$ (three times)
			$\cup\{\langle i+D_3\rangle \bmod 31, i\in I(5)\}$ (twice).
31	6	12	$X=Z(31,3)$, $G=\mathrm{AF}(31)$,
			$D_0=\langle 0,1,7\rangle$, $D_1=\langle 0,1,13\rangle$, $D_2=\langle 0,2,10\rangle$,
			$D_3=\langle 0,2,11\rangle$, $D_4=\langle 0,3,10\rangle$, $D_5=\langle 6I(5)\rangle$,
			$\mathscr{B}=\{\langle i+D_j, i+15+D_j\rangle \bmod 31, i\in I(15), j\in I(5)\}$
			$\cup\{\langle\emptyset, i+D_5\rangle \bmod 31, i\in I(6)\}$ (twice).
31	7	35	Lemma 3.6, $32\in B_3(8,7)$—below.
31	8	168	Lemma 3.7, $30\in B_3(7,30)\cap B_3(8,138)$—above.
31	9	84	Lemma 3.6, $32\in B_3(10,12)$—below.
31	10	24	Unknown.
31	10	48	$X=Z(31,3)$, $G=\mathrm{AF}(31)$,
			$D_0=\langle 3I(10)\rangle$, $D_1=\langle 10I(3)\rangle$, $D_2=\langle 0,1,3,9,14\rangle$,
			$D_3=\langle 0,1,3,9,11\rangle$, $D_4=\langle 0,1,5,6,9\rangle$,
			$\mathscr{B}=\{\langle i+D_0\rangle \bmod 31, i\in I(3)\}\cup\{\langle\emptyset, i+D_1, i+5+D_1, i+7+D_1\rangle \bmod 31, i\in I(10)\}$
			$\cup\{\langle i+D_j, i+15+D_j\rangle \bmod 31, i\in I(15), j=2,3,4\}$.
31	10	72	$X=Z(31,3)$, $G=\mathrm{AF}(31)$,
			$D_0=\langle 6I(5)\rangle$, $D_1=\langle 0,1,6,9,12\rangle$,
			$D_2=\langle 0,1,2,9,10\rangle$, $D_3=\langle 0,1,2,6,9\rangle$,
			$\mathscr{B}=\{\langle i+D_0, i+j+1+D_0\rangle \bmod 31, i\in I(6), j\in I(2)\}$
			$\cup\{\langle i+D_1, i+15+D_1\rangle \bmod 31, i\in I(15)\}$
			$\cup\{\langle i+D_j, i+15+D_j\rangle \bmod 31, i\in I(15), j=2,3\}$ (twice).
31	11	99	$X=Z(31,3)$, $G=\mathrm{AF}(31)$,
			$D_0=\langle 0,1,3,6,11\rangle$, $D_{1+j}=\langle 0,1,5+j,10-2j,12\rangle$, $j\in I(2)$.
			$D_3=\langle 0,1,2,10,13\rangle$, $D_4=\langle 3I(10)\rangle$,
			$\mathscr{B}=\{\langle\emptyset, i+D_0, i+15+D_0\rangle \bmod 31, i\in I(15)\}$ (twice)
			$\cup\{\langle\emptyset, i+D_j, i+15+D_j\rangle \bmod 31, i\in I(15), j\in N(3)\}$
			$\cup\{\langle\emptyset, i+D_4\rangle \bmod 31, i\in I(3)$ (four times).
31	12	220	$X=Z(31,3)$, $G=\mathrm{AF}(31)$,
			$D_0=\langle 0,1,3,5,7,10\rangle$, $D_{1+j}=\langle 0,1,2,5,8+j,11\rangle$, $j\in I(2)$,
			$D_{3+j}=\langle 0,1,2,6+j,10-2i,13\rangle$, $j\in I(2)$,
			$D_5=\langle 0,1,2,5,6,8\rangle$, $D_6=\langle 0,1,2,3\rangle$,
			$\mathscr{B}=\{\langle i+D_0, i+15+D_0\rangle \bmod 31, i\in I(15)>$(four times)
			$\cup\{\langle i+D_j, i+15+D_j\rangle \bmod 31, i\in I(15), j\in N(5)\}$
			$\cup\{\langle i+D_6, i+10+D_6, i+20+D_6\rangle \bmod 31, i\in I(10)\}$.
31	13	286	$X=Z(31,3)$, $G=\mathrm{AF}(31)$,
			$D_j=\langle 0,1,3,4+3j,7+3j,9+3j\rangle$, $j\in I(2)$,
			$D_2=\langle 0,1,2,3,4,11\rangle$, $D_{3+j}=\langle 0,1,2+j,6,10+j,11+j\rangle$ $j\in I(2)$,
			$D_5=\langle 0,1,2,5,6,8\rangle$, $D_6=\langle 0,1,3,7\rangle$,
			$\mathscr{B}=\{\langle\emptyset, i+D_j, i+15+D_j\rangle \bmod 31, i\in I(15), j\in I(3)\}$ (twice)
			$\cup\{\langle\emptyset, i+D_j, i+15+D_j\rangle \bmod 31, i\in I(15), j=3,4,5\}$
			$\cup\{\langle\emptyset, i+D_6, i+10+D_6, i+20+D_6\rangle \bmod 31, i\in I(10)\}$.

Table 3.18. (Contd.)

v	k	λ	Construction
32	14	1092	Lemma 3.5, $31 \in B_3(15, 91)$—below, $15 \in B_3(14, 12)$—Lemma 3.1.
31	15	91	Lemma 3.6, $32 \in B_3(16, 7)$—below.
32	5	2	$X = \mathrm{GF}(32, x^5 = x^2 + 1)$, $G = \mathrm{AF}(32)$, $D_0 = \langle 0, 1, 5, 21, 23\rangle$, $\mathcal{B} = \{\langle i + D_0\rangle \bmod 32, i \in I(31)\}$.
32	6	2	Unknown.
32	6	4	$X = \mathrm{GF}(32, x^5 = x^3 + 1)$, $G = \mathrm{AF}(32)$, $D_0 = \langle 0, 1, 2, 3, 6, 9\rangle$, $\mathcal{B} = \{\langle i + D_0\rangle \bmod 32, i \in I(31)\}$.
32	6	6	$X = \mathrm{GF}(32, x^5 = x^3 + 1)$, $G = \mathrm{AF}(32)$, $D_0 = \langle 0, 2, 4, 25, 28\rangle$, $D_1 = \langle 0, 1, 2, 3, 4, 14\rangle$, $\mathcal{B} = \{\langle\langle a\langle 4\rangle \oplus b\langle 3\rangle \oplus c\langle 2\rangle \oplus d\langle 1\rangle \oplus \langle \emptyset, D_0\rangle\rangle + i\rangle, a, b, c, d \in I(2), i \in I(31)\}$ $\cup \{\langle i + D_1\rangle \bmod 32, i \in I(31)\}$.
32	7	7	$X = \mathrm{GF}(32, x^5 = x^2 + 1)$, $G = \mathrm{AF}(32)$, $D_0 = \langle 0, 1, 2, 4, 8, 16\rangle$, $\mathcal{B} = \{\langle \emptyset, i + D_0\rangle \bmod 32, i \in I(31)\}$.
32	8	7	$X = \mathrm{GF}(32, x^5 = x^3 + 1)$, $G = \mathrm{AF}(32)$, $D_0 = \langle 0, 6, 10, 13, 15, 22, 30\rangle$, $D_1 = \langle 0, 1, 2, 14, 15, 22, 28\rangle$, $D_2 = \langle 1, 10, 22, 25, 27, 28, 30\rangle$, $D_3 = \langle 1, 2, 10, 13, 15, 21, 25\rangle$, $D_4 = \langle 0, 2, 6, 10, 21, 27, 28\rangle$, $\mathcal{B} = \{\langle\langle a\langle 3 + j\rangle \oplus b\langle 2 + j\rangle \oplus \langle \emptyset, D_j\rangle\rangle + i\rangle, a, b \in I(2), j \in I(2), i \in (31)\}$ $\cup \{\langle\langle a\langle 3\rangle \oplus b\langle 0\rangle \oplus \langle \emptyset, D_j\rangle\rangle + i\rangle, a, b \in I(2), i \in I(31), j = 2, 3\}$ $\cup \{\langle\langle a\langle 3\rangle \oplus b\langle 1\rangle \oplus \langle \emptyset, D_4\rangle\rangle + i\rangle, a, b \in I(2), i \in I(31)\}$.
32	9	84	Lemma 3.5, $32 \in B_3(10, 12)$—below, $10 \in B_3(9, 7)$—Lemma 3.1.
32	10	12	$X = \mathrm{GF}(32, x^5 = x^3 + 1)$, $G = \mathrm{AF}(32)$, $D_0 = \langle 0, 6, 9, 10, 13, 24, 26, 29, 30\rangle$, $\mathcal{B} = \{\langle\langle a\langle 4\rangle \oplus b\langle 3\rangle \oplus c\langle 2\rangle \oplus d\langle 1\rangle \oplus \langle \emptyset, D_0\rangle\rangle + i\rangle, a, b, c, d \in I(2), i \in I(31)\}$.
32	11	33	Unknown.
32	11	66	$X = \mathrm{GF}(32, x^5 = x^3 + 1)$, $G = \mathrm{AF}(32)$, $D_0 = \langle 0, 1, 2, 5, 7, 8, 10, 13, 17, 25, 29\rangle$, $D_1 = \langle 0, 1, 2, 5, 7, 8, 10, 13, 18, 19, 26\rangle$, $\mathcal{B} = \{\langle i + D_j\rangle \bmod 32, i \in I(31), j \in I(2)\}$.
32	11	99	Theorem 3.13.
32	12	11	Unknown.
32	12	22	$X = Z(31, 3) \cup \{\infty\}$, $G = \mathrm{AF}(31)$, $D_0 = \langle 6I(5)\rangle$, $D_1 = \langle 0, 1, 4, 5\rangle$, $\mathcal{B} = \{\infty, \emptyset, i + D_0, i + 1 + D_0\rangle \bmod 31, i \in I(6)\}$ $\cup \{\langle i + D_1, i + 10 + D_1, i + 20 + D_1\rangle \bmod 31, i \in I(10)\}$.
32	12	33	Unknown.
32	12	55	$X = \mathrm{GF}(32, x^5 = x^3 + 1)$, $G = \mathrm{AF}(32)$, $D_0 = \langle 0, 1, 3, 4, 5, 6, 8, 10, 18, 19, 26\rangle$, $D_1 = \langle 0, 1, 3, 5, 6, 8, 10, 17, 19, 25, 26\rangle$, $D_2 = \langle 0, 1, 3, 6, 10, 17, 18, 19, 25, 26, 29\rangle$, $D_3 = \langle 0, 1, 2, 3, 5, 7, 14, 22, 23, 26, 29\rangle$, $D_4 = \langle 0, 1, 2, 3, 5, 7, 11, 15, 22, 23, 26\rangle$, $\mathcal{B} = \{\langle\langle a\langle 2\rangle \oplus b\langle 1\rangle \oplus c\langle 0\rangle \oplus \langle \emptyset, D_j\rangle\rangle + i\rangle, a, b, c \in I(2), i \in I(31), j \in I(3)\}$ $\cup \{\langle\langle a\langle 4\rangle \oplus b\langle 2\rangle \oplus c\langle 0\rangle \oplus \langle \emptyset, D_j\rangle\rangle + i\rangle, a, b, c \in I(2), i \in I(31), j = 3, 4\}$.
32	13	286	$X = Z(31, 3) \cup \{\infty\}$, $G = \mathrm{AF}(31)$, $D_0 = \langle 0, 1, 2, 3, 7 + j, 11\rangle$, $j \in I(2)$, $D_2 = \langle 5I(6)\rangle$, $D_3 = \langle 0, 1, 2, 3, 4, 11\rangle$, $D_{4+j} = \langle 0, 1, 2, 4, 3 + 2j, 13 - 6j\rangle$, $j \in I(2)$, $D_{6+j} = \langle 0, 1, 2, 3, 5 + 4j, 10\rangle$, $j \in I(2)$, $\mathcal{B} = \{\langle \infty, i + D_0, i + 15 + D_0\rangle \bmod 31, i \in I(15)\}$ $\cup \{\langle \infty, i + D_1, i + 15 + D_1\rangle \bmod 31, i \in I(15)\}$ (three times)

Table 3.13. (Contd.)

v	k	λ	Construction
			$\cup\{\langle\infty, i+D_2, i+1+D_2\rangle \bmod 31, i\in I(5)\}$
			$\cup\{\langle\emptyset, i+D_3, i+15+D_3\rangle \bmod 31, i\in I(15)\}$ (twice)
			$\cup\{\langle\emptyset, i+D_j, i+15+D_j\rangle \bmod 31, i\in I(15), j=4,5,6,7\}$
			$\cup\{\langle\emptyset, i+D_2, i+1+D_2\rangle \bmod 31, i\in I(5)\}$.
32	14	182	$X=Z(31,3)\cup\{\infty\}$, $G=\mathrm{AF}(31)$,
			$D_0=\langle 0,1,5,7,9,11\rangle$, $D_1=\langle 5I(6)\rangle$,
			$D_{2+j}=\langle 0,1,2,3,5,9,6+6j\rangle$, $j\in I(2)$,
			$D_4=\langle 0,1,2,4,6,11,12\rangle$,
			$\mathscr{B}=\{\langle\infty,\emptyset, i+D_0, i+15+D_0\rangle \bmod 31, i\in I(15)\}$ (twice)
			$\cup\{\langle\infty,\emptyset, i+D_1, i+2+D_1\rangle \bmod 31, i\in I(5)\}$
			$\cup\{\langle i+D_j, i+15+D_j\rangle \bmod 31, i\in I(15), j=2,3,4\}$.
32	15	91	Lemma 3.15.
32	16	7	Theorem 3.10.

Table 4.1. Possible values of $N(3,k,v)$ for $5\leq k\leq\frac{1}{2}v$, $v\leq 32$

v \ k	5	6	7	8	9	10	11	12	13	14	15	16
10	0	—	—	—	—	—	—	—	—	—	—	—
11	2	—	—	—	—	—	—	—	—	—	—	—
12	0	0	—	—	—	—	—	—	—	—	—	—
13	0	0	—	—	—	—	—	—	—	—	—	—
14	0	0	0	—	—	—	—	—	—	—	—	—
15	0	0	0	—	—	—	—	—	—	—	—	—
16	0	2	0, 5	0	—	—	—	—	—	—	—	—
17	0	0	0, 7	0	—	—	—	—	—	—	—	—
18	0	0	0	0	0	—	—	—	—	—	—	—
19	0	0	0	0	0	—	—	—	—	—	—	—
20	0, 2	0	0	0	0	0	—	—	—	—	—	—
21	0	0, 4	0, 15, 30, 75	0, 84	0	0	—	—	—	—	—	—
22	0	0	2, 3, 7	0, 6	0	0, 6, 12 18, 24, 30, 36 60, 66, 78	0	—	—	—	—	—
23	0	0	0	0, 8	0, 12	0	0, 15 30	—	—	—	—	—
24	0	0	0	0	0	0	0	0	—	—	—	—
25	0	0	0	0	0, 21	0, 24	0	0	—	—	—	—
26	0	0	0	0, 7	0, 21	*	0, 33	0	0	—	—	—
27	0	0	0	0	0	0	0	0	0	—	—	—
28	0	0	0, 5	0	0	0, 22	0	0	0, 22, 44, 88, 154	0	—	—
29	0	0	0, 5	0, 4, 12, 20	0, 14	0	0	0	0	0, 52	—	—
30	0	0	0	0, 6, 12, 18, 24, 30, 36	0, 6, 12, 18, 24	0	0	0	0	0, 39	0	—
31	0	0, 4	0	0	0	0, 24	0	0	0	0	0	—
32	0	0, 2	0	0	0	0	0, 33	0, 11, 33	0	0	0	0

$*=\{3,6,9,12,15,18,21,27,30,33,39,42,51,54,57,63,66,75,78,87,99,102,111,123,147\}$

4. Conclusion

We conclude with a list of possible values for the function $N(3, k, v)$, $5 \leq k \leq \frac{1}{2}v$, $v \leq 32$, based on the results of Sections 2 and 3. If more than one possible value is given this indicates that the existence of some design is undecided. (See Table 4.1.)

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